Twice the Angle

- Circle Theorems 3: Angle at the Centre Theorem -

Definitions

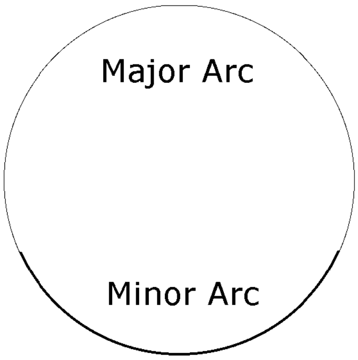
An <u>arc</u> of a circle is a contiguous (i.e. no gaps) portion of the circumference.

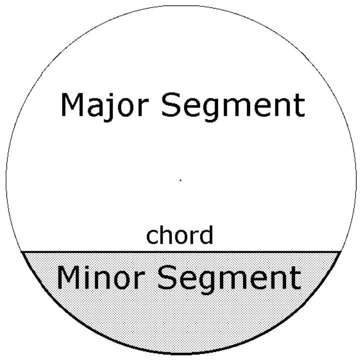
An arc which is half of a circle is called a <u>semi-circle</u>.

An arc which is shorter than a semi-circle is called a <u>minor arc</u>.

An arc which is greater than a semi-circle is called a <u>major arc</u>.

Clearly, for every minor arc there is a corresponding major arc.





A <u>segment</u> of a circle is a figure bounded by an arc and its chord.

If the arc is a minor arc then the segment is a <u>minor segment</u>.

If the arc is a major arc then the segment is a <u>major segment</u>.

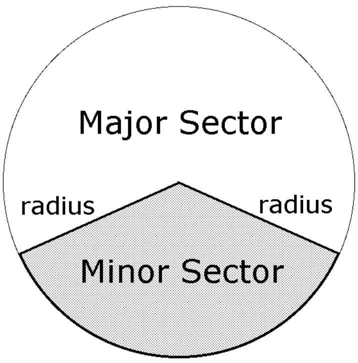
Clearly, for every minor segment there is a corresponding major segment.

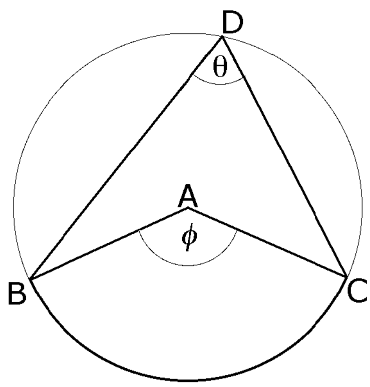
A <u>sector</u> of a circle is a figure bounded by two radii and the included arc.

If the arc is a minor arc then the sector is a <u>minor sector</u>.

If the arc is a major arc then the sector is a <u>major sector</u>.

Clearly, for every minor sector there is a corresponding major sector.





The word <u>subtend</u> means to hold up or support. For example, the minor arc BC subtends an angle θ ($\angle$BDC) at a point D on the major arc.

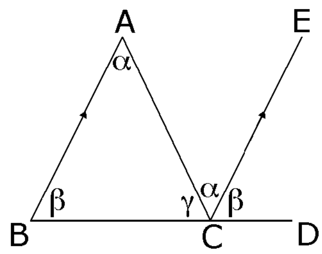
We could also have said that the chord BC subtends $\angle$BDC at the point D.

Similarly, the chord or minor arc BC subtends $\angle$BAC at the centre A. The $\angle$BAC is called a central angle and is sometimes measured by the length of the minor arc.

Twice the Angle

Prior Learning

Theorem: <u>The exterior angle of any triangle is equal to the sum of the interior opposite angles.</u>



Given: any $\triangle$ABC with BC extended to D

To Prove: m$\angle$ACD = m$\angle$BAC + m$\angle$ABC

Construction: Draw CE // BA

Proof: m$\angle$ACE = m$\angle$BAC = α (alternate $\angle$s: CE // BA)

m$\angle$ECD = m$\angle$ABC = β (corresponding $\angle$s: CE // BA)

adding $\Rightarrow$ m$\angle$ACD = m$\angle$BAC + m$\angle$ABC = $\alpha + \beta$ **QED**

Corollary: <u>The sum of the angles of any triangle is 180°</u>

$\triangle$ABC is any triangle

Since $\alpha + \beta + \gamma$ measure adjacent angles on the line BCD

It follows that $\alpha + \beta + \gamma = 180°$.

But the sum of the angles of the $\triangle$ABC is also $\alpha + \beta + \gamma$

$\Rightarrow$ The sum of the angles of any triangle is 180°

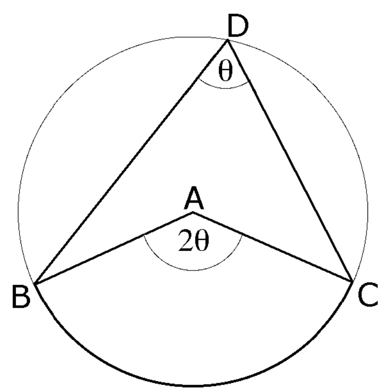
Purpose

To demonstrate the Angle at the Centre of a Circle Theorem

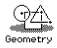
from Book III of Euclid's *Elements*

Angle at the Centre Theorem

<u>The angle which an arc of a circle subtends at the centre of a circle is double the angle which it subtends at any point on the remaining part of the circumference.</u>



ClassPad Time

Switch on your ClassPad and tap on the Geometry icon .

If your worksheet has data from a previous investigation, check if you need to save this before you clear it from the screen.

Open the file **ChordSt**.

From the Draw Menu tap on Point.

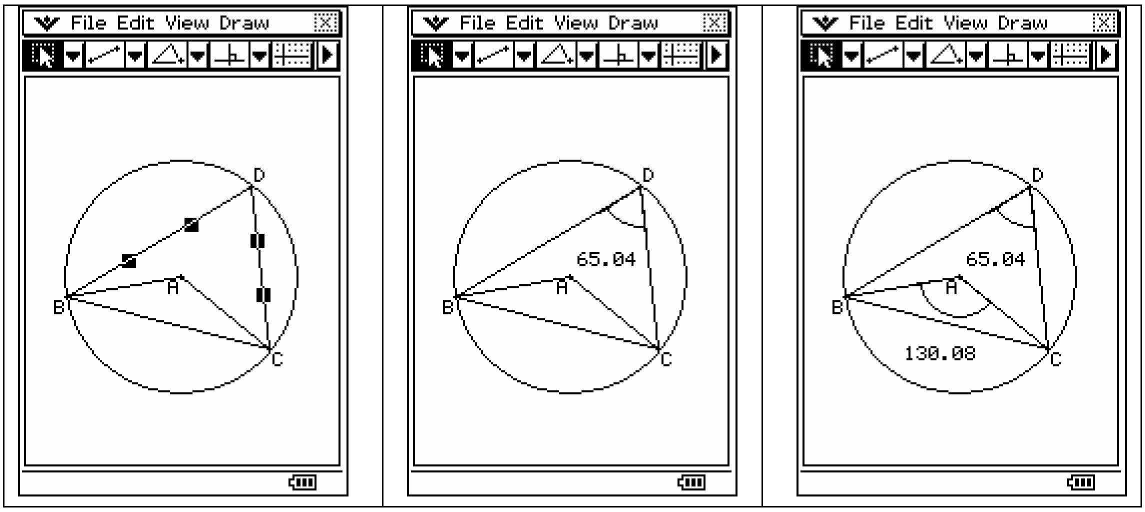
Tap once somewhere on the major arc BC.

This point will be automatically labelled D.

From the View Menu tap on Select or

get the Selection Arrow  from the Tool Bar.

Highlight the line segments BD and CD.



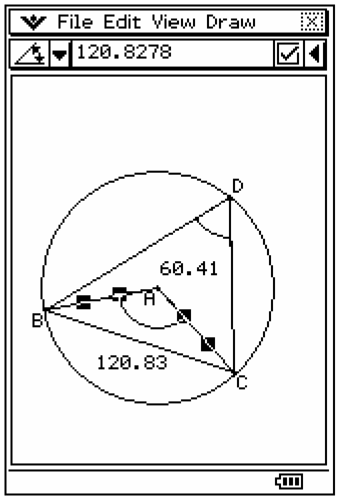
From the Draw Menu tap on Attach Angle. Now you should find that ClassPad has measured ∠BDC for you.

Tap on the radius BA and on the radius CA and both radii should be highlighted. From the Draw Menu tap on Attach Angle. Now you should find that ClassPad has measured ∠BAC for you.

Form a hypothesis relating m∠BDC and m∠BAC and write it in this space.

Checkpoint 

Checking the hypothesis



Try moving the point C. At each position of C around the circle you should find that m∠BDC = 2 × m∠BAC. Sometimes you get diagrams, like this one, where m∠BDC doesn't look exactly double m∠BAC.

From the View Menu tap on Select or get the Selection Arrow from the Tool Bar.
Highlight the radii AB and AC.
Then tap on the arrow ▶ at the far right of the Tool Bar. This takes you "around the corner" to the Measurement Bar. ClassPad has been programmed to anticipate that you wanted to know the size of the angle between the line segments you have highlighted. You can now see that any discrepancy is only because ClassPad is rounding the size of the angle to two decimal places.

Animation

Tap in free space to clear any selections.
From the View Menu tap on Select and use the Selection Arrow to highlight the point C and also highlight the circle.
From the Edit Menu choose Animate and Add Animation.
From the Edit Menu choose Animate and Go (repeat).
As the point C moves around the circle you will find that m∠BDC = 2 × m∠BAC. **Save your work as Centre1**.

Checkpoint

Deductive Reasoning

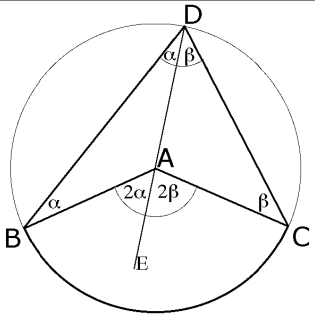


Figure 1

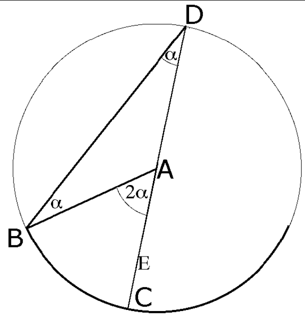


Figure 2

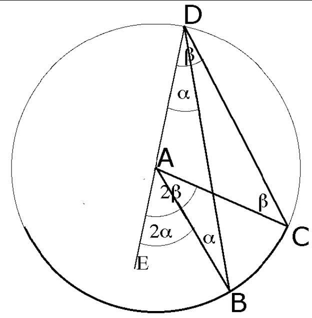


Figure 3

Given: Circle centre A in which the arc BC subtends a central angle BAC and an angle BDC at a point D on the circumference.

To Prove:

m∠BAC = $2 \times$ m∠BDC

Construction:

Draw DA extended to E

Proof: △BAD is isosceles (AB = AD: radii)

⇒ m∠ABD = m∠ADB = α (base∠s isos△)

⇒ m∠BAE = 2α (Ext Angle Thm)

Similarly m∠CAE = 2β

In fig 1,

m∠BAC = $(2\alpha + 2\beta) = 2(\alpha + \beta) = 2 \times$ m∠BDC

In fig 2,

m∠BAC = $(2\alpha) = 2(\alpha) = 2 \times$ m∠BDC

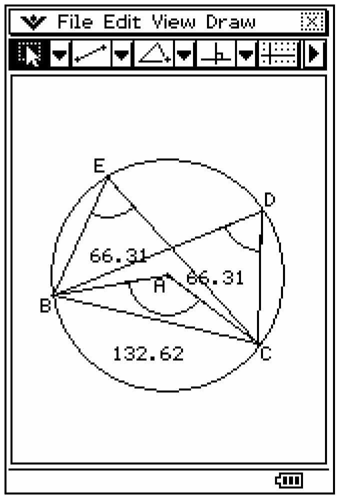
In fig 3,

m∠BAC = $(2\beta - 2\alpha) = 2(\beta - \alpha) = 2 \times$ m∠BDC

QED

A Corollary of the Angle at the Centre Theorem

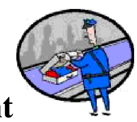
If you move the point D to any point on the major arc you find that it is always half the angle at the centre. Therefore, if you set up another angle at a point E on the same arc, this new angle must also be half the angle at the centre. If they are both half the angle at the centre, they have to be equal to each other.



Now it's time to fly solo.
Set up a new angle BEC, measure it and show that m∠BEC is always equal to m∠BEC.
Animate this feature and then show your teacher.

Finally, everything said about angles in the major segment is also true for angles in the minor segment.

Checkpoint



Checkpoints

Checkpoint 1

We are looking for a restatement of the Theorem as symbols

m∠BAC = $2 \times$ m∠BDC

or as words.

Checkpoint 2

This provides a good opportunity to talk about the difference in accuracy between an attached angle and the measurement displayed in the Measurement Bar.

Checkpoint 3

We are looking to see if one of the points is animated and the angles at D and E remain equal.