Describing Change
Describing Change

Version 1.00 – October 2008

Written by Anthony Harradine and Alastair Lupton

Copyright © Harradine and Lupton 2008.

Copyright Information.

The materials within, in their present form, can be used free of charge for the purpose of facilitating the learning of children in such a way that no monetary profit is made.

The materials cannot be used or reproduced in any other publications or for use in any other way without the express permission of the authors.
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Managing a gas field.</td>
<td>5</td>
</tr>
<tr>
<td>2. Things that stack.</td>
<td>6</td>
</tr>
<tr>
<td>3. Talking about change.</td>
<td>7</td>
</tr>
<tr>
<td>4. Polygons and Angles.</td>
<td>9</td>
</tr>
<tr>
<td>5. Modelling Deterministic Change – 1.</td>
<td>11</td>
</tr>
<tr>
<td>6. Signal and Noise.</td>
<td>13</td>
</tr>
<tr>
<td>7. Modelling Stochastic Change – 1.</td>
<td>17</td>
</tr>
<tr>
<td>8. Invest smarter.</td>
<td>21</td>
</tr>
<tr>
<td>9. Simple exponential functions.</td>
<td>23</td>
</tr>
<tr>
<td>10. Modelling Deterministic Change – 2.</td>
<td>24</td>
</tr>
<tr>
<td>11. Modelling Stochastic Change – 2.</td>
<td>277</td>
</tr>
<tr>
<td>12. eTech Support.</td>
<td>31</td>
</tr>
<tr>
<td>13. Answers.</td>
<td>??</td>
</tr>
</tbody>
</table>
Using this resource.

This resource is not a text book.

It contains material that is hoped will be covered as a dialogue between students and teacher and/or students and students.

You, as a teacher, must plan carefully ‘your performance’. The inclusion of all the ‘stuff’ is to support:
- you (the teacher) in how to plan your performance – what questions to ask, when and so on,
- the student that may be absent,
- parents or tutors who may be unfamiliar with the way in which this approach unfolds.

Professional development sessions in how to deliver this approach are available.

Please contact

The Noel Baker Centre for School Mathematics
Prince Alfred College,
PO Box 571 Kent Town, South Australia, 5071
Ph. +61 8 8334 1807
Email: aharradine@pac.edu.au

Legend.

**EAT – Explore And Think.**

These provide an opportunity for an insight into an activity from which mathematics will emerge – but don’t pre-empt it, just explore and think!

At certain points the learning process should have generated some **burning mathematical questions** that should be discussed and pondered, and then answered as you learn more!

**Time to Formalise.**

These notes document the learning that has occurred to this point, using a degree of formal mathematical language and notation.

**Examples.**

Illustrations of the mathematics at hand, used to answer questions.
1. Managing a gas field.

There is a great deal of planning involved in running a gas production site such as the one pictured.

Once a site has been chosen and gas wells have been drilled, productivity is monitored by measuring the rate of flow of the gas out of each well.

The scenario (a real one)
A gas production site in central Australia contains, potentially, up to six wells. At five of these, wells are already installed and producing gas.

After considering demand levels and production costs, the Reservoir Engineer decides that, for the site to be considered viable, the average daily rate of flow from the entire site in any given month must be at least 5 MMscf/day (millions of cubic feet per day). If the average daily rate for a given month is expected to fall below this, the sixth well will be installed to increase gas production.

The table below gives the actual average daily flow rate from the site for the months shown. During this period only five wells are installed and producing gas.

<table>
<thead>
<tr>
<th>Month (end date)</th>
<th>Relative time t (months)</th>
<th>Rate of Gas Flow f (MMscf/d)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5/31/1998</td>
<td></td>
<td>51.717</td>
</tr>
<tr>
<td>6/30/1998</td>
<td></td>
<td>47.724</td>
</tr>
<tr>
<td>7/31/1998</td>
<td></td>
<td>36.717</td>
</tr>
<tr>
<td>8/31/1998</td>
<td></td>
<td>31.755</td>
</tr>
<tr>
<td>9/30/1998</td>
<td></td>
<td>28.066</td>
</tr>
<tr>
<td>10/31/1998</td>
<td></td>
<td>22.248</td>
</tr>
<tr>
<td>11/30/1998</td>
<td></td>
<td>22.199</td>
</tr>
<tr>
<td>12/31/1998</td>
<td></td>
<td>19.154</td>
</tr>
<tr>
<td>1/31/1999</td>
<td></td>
<td>16.377</td>
</tr>
<tr>
<td>2/28/1999</td>
<td></td>
<td>14.611</td>
</tr>
<tr>
<td>3/31/1999</td>
<td></td>
<td>13.403</td>
</tr>
<tr>
<td>4/30/1999</td>
<td></td>
<td>12.72</td>
</tr>
<tr>
<td>5/31/1999</td>
<td></td>
<td>11.285</td>
</tr>
</tbody>
</table>

EAT 1

Based on the information provided above, estimate the month in which the Reservoir Engineer will require a sixth well in order to increase gas production. In what ways can you use the data set provided to support your estimate?
2. **Things that stack.**

Have you ever seen a Cup Snake (sometimes known as a Beer Cup Snake)?

 Whilst their preferred habitat is international cricketing fixtures, this photo shows a well-documented sighting at a recent rock concert\(^1\)

![Foo Fighters concert, Wembley Stadium, June 6\textsuperscript{th} 2008 (the cup snake appeared during the performance of the support act, Australian band Supergrass. )]

---

**EAT 2 A**

Obtain 2 disposable cups. By studying only these two cups, estimate the length in centimetres (\(L\)) of the cup snake made from

\[
100, 200, 300, 400, 500, 600, \ldots n \text{ such cups.}
\]

Explain how you obtained these estimates.

---

**EAT 2 B**

With the help of your classmates, construct a cup snake in such a way as to enable you to check your estimate for the length of a cup snake containing 100, 200, 300, 400, 500 and 600 cups.

Study relationship between your estimated lengths and the measured lengths.

In the light of what you observe, revisit your estimate for the length of a cup snake containing \(n\) such cups.

---

Whilst the length of cup snakes may not be of a great deal of practical significance, a number of objects are stored in stacked form. For those who store such objects, the length or height of such stacks is an important part of this storage.

---

\(^1\) Foo Fighters concert, Wembley Stadium, June 6\textsuperscript{th} 2008 (the cup snake appeared during the performance of the support act, Australian band Supergrass. )
3. **Talking about change.**

Change can be observed in most of the systems that make up the world around us, in simple systems like the way the balance of an investment account changes over time, as well as in more complex systems like the growth of plants in response to environmental factors like the supply of water and nutrients.

Human-designed systems, like the investment account, often change in regular and predictable ways, allowing us to model these systems with a high degree of accuracy. Change in non human-designed systems, like the growth of plants, can be somewhat inconsistent. As a result, the modelling of such systems is harder and our ability to predict their behaviour is more questionable.

**More specifically**

It is a quantity associated with a system that changes (i.e. value in $, weight in kg etc). Hence we could say that it is the quantity that shows variation. The variation in one quantity can often be explained by the variation in another. Consider the following:

To what degree can

...diamond price be explained by its weight?

...the incidence of lung cancer be explained by degree of exposure to cigarette smoke?

...global mean surface temperature be explained by the amount of atmospheric carbon dioxide?

These questions suggest the need to explore the relationship between two quantities or variables. We can describe this relationship in two ways.

It is likely that, in general,

- variation in diamond price is in **response** to variation in its weight.
- variation in the incidence of lung cancer is in **response** to variation in exposure to cigarette smoke.

Or to put that another way, to some degree

- weight **explains** the variation in diamond price.
- exposure to cigarette smoke **explains** the incidence of lung cancer.
By thinking in this way we can see that there are two different roles played by the variables in these situations. These roles can be categorised as either 

(1) **Response** or (2) **Explanatory**

In a situation, in order to determine which variable plays which role, ask yourself

*Which of the variables explains the variation that occurs in the other variable?*

For example, does diamond price explain the variation in weight or does weight explain the variation in diamond price?2

In the previous three examples the following is clear,

<table>
<thead>
<tr>
<th>Response Variable</th>
<th>Explanatory Variable</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diamond Price</td>
<td>weight</td>
</tr>
<tr>
<td>incidence of lung cancer</td>
<td>degree of exposure to cigarette smoke</td>
</tr>
<tr>
<td>global mean surface temperature</td>
<td>amount of atmospheric carbon dioxide</td>
</tr>
</tbody>
</table>

In studies, in laboratories and elsewhere, the idea is to make orderly changes to the explanatory variable and observe the variation in the response variable.

For example, nutrient concentration can be changed in an orderly way (i.e. increased by specific amounts) and the effect on the height of a plant can be observed.

Sometimes, the response variable is referred to as the dependant variable, as the value it takes tends to depend on the independent variable, which is another name for the explanatory variable.

We have used the term explain not cause. The cause of the variation may be something quite different to the explanatory variable.

For example, the price of a new Commodore may be explained by time but what caused the variation?

Medical Researchers develop a new drug. They need to find the dose at which it is most effective. The drug is developed to reduce the level of histamines in the body.

What variable are involved and what is their role?

---

2 More often than not the response variable is the “interesting variable” and the explanatory variable is the “boring variable”.

© A Harradine & A Lupton  Draft, Oct 2008 WIP
4. Polygons and Angles.

A polygon is a closed figure with \( n \) sides. Each and every polygon with a specific number of sides has a set sum of its interior angles, as shown in the table below:

<table>
<thead>
<tr>
<th>number of sides ( n )</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>angle sum ( A )</td>
<td>180</td>
<td>360</td>
<td>540</td>
<td>720</td>
<td>900</td>
<td>1080</td>
</tr>
</tbody>
</table>

By thinking about how the values of \( n \) and \( A \) change, this table can be understood in the following way.

<table>
<thead>
<tr>
<th>consecutive adders in ( n )</th>
<th>+1</th>
<th>+1</th>
<th>+1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n )</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>( A )</td>
<td>180</td>
<td>180+180</td>
<td>180+180+180</td>
</tr>
<tr>
<td>consecutive adders in ( A )</td>
<td>+180</td>
<td>+180</td>
<td>+180</td>
</tr>
</tbody>
</table>

For a constant additive change in the explanatory variable, a constant additive change occurs in the response variable.

Change that can be described in this way is referred to as linear. A graphic model of this situation looks as follows:

Would it make sense to join these dots?
Note: A useful convention is that the explanatory variable is represented horizontally and the response variable is represented vertically. This system, the relationship between the interior Angle Sum \((A)\) and the number of sides \((n)\) of a polygon, can be described algebraically.

Members of the family of linear functions\(^3\) \(y = mx + c\) can be used to describe systems where a constant additive change in the explanatory variable results in a constant additive change in the response variable, as seen below:

<table>
<thead>
<tr>
<th>consecutive adders in (x)</th>
<th>+1</th>
<th>+1</th>
<th>1</th>
<th>...</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x)</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>...</td>
</tr>
<tr>
<td>(y)</td>
<td>c</td>
<td>(m + c)</td>
<td>(2m + c)</td>
<td>(3m + c)</td>
<td>...</td>
</tr>
<tr>
<td>consecutive adders in (y)</td>
<td>(m)</td>
<td>(m)</td>
<td>(m)</td>
<td>...</td>
<td>(m)</td>
</tr>
</tbody>
</table>

\[x \vdash y = \sum (x) = (x) = (x)\] or \[m \vdash y = \sum (x) = (x) = (x)\]

4.1 Can you use the knowledge?

1. Write down the linear function that describes the relationship between the interior Angle Sum \((A)\) and the number of sides \((n)\) of a polygon.

2. a. What is the slope / gradient of the line joining the points on the graph of \(A\) against \(n\) on the previous page?
   b. What does the slope / gradient represent in the context of system?

3. Explain the meaning of the vertical axis intercept in this context.

4. Use your linear function to determine the angle sum of a dodecagon.

5. Use your linear function to show that no polygon has an angle sum of \(1680^\circ\).

6. Through geometric reasoning, it is possible deduce the relationship that we have obtained between the interior angles and the number of sides of a polygon. See if you can do so (a hint is provided in the Answers section).

---

\(^3\) A function is a mathematical rule that operates on a value (the “input” - often \(x\)) to obtain a unique value (the “output” - commonly \(y\)). As such, it is a rule for obtaining \(y\) by operating on \(x\), sometimes referred to a rule “for \(y\) in terms of \(x\)”. 
5. **Modelling Deterministic Change – 1.**

Systems, like the polygon’s internal angle sum, that change in a predictable or non-random way are called deterministic systems. If a system’s change contains a degree of randomness or a limited degree of predictability then it is called a stochastic system. We are now going to work with some more deterministic systems.

5.1 **Simple Interest.**

In non-commercial loans (i.e. amongst family members) simple interest is often paid by the borrower to the lender. This is a percentage of the borrowed amount, charged per time period (i.e. per year) of the loan.

Jonathan borrows $5 400 from his father to purchase his first car. To compensate him, Jonathan he agrees to pay him 4% p.a. (per annum or per year) in addition to the borrowed amount (when he can afford to repay the loan).

To study the relationship between the length of the loan ($t$ years) and the amount owed ($A$) then

1. Define the role of the two variables.
2. Complete the following table

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3. Explain why the relationship between these two variables is linear.
4. Represent this relationship graphically.
5. Write down the relationship between $A$ and $t$ algebraically. (in other words, write down a function of $A$ in terms of $t$).
6. Use this function to determine the amount that Jonathan owes his father if he repays the loan
   a. after 10 years  
   b. after 7 years and 9 months.
7. How long will the loan have been outstanding at the time when Jonathan owes his father $9 000?

5.2 **International Standard (SI) units – Speed.**

Speed can be measured in a number of different units. Whilst in science the SI unit metres per second is preferred, kilometres per hour is more widely understood. Consider the following table,
<table>
<thead>
<tr>
<th>$M$ (metres per second)</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K$ (kilometres per hour)</td>
<td>36</td>
<td>72</td>
<td>108</td>
<td>144</td>
<td>180</td>
</tr>
</tbody>
</table>

1. Explain why the relationship between $M$ and $K$ is a linear one.

2. Write down a function for $K$ in terms of $M$.

3. Use this function to convert
   a. $18.6 \text{ m/s}$ to $\text{km/h}$.
   b. $1200 \text{ m/s}$ to $\text{km/h}$.
   c. $60 \text{ km/h}$ to $\text{m/s}$.
   d. $155 \text{ km/h}$ to $\text{m/s}$.

5.3 International Standard (SI) units – Temperature.

Whilst most of the world exclusively use SI units (also known as metric units), USA and Great Britain still frequently work with customary units (a.k.a. imperial units). When obtaining culinary and meteorological information sourced from these countries, temperatures are commonly given in degrees Fahrenheit. Some conversions to degrees Celsius (the SI unit of temperature) are provided below.

<table>
<thead>
<tr>
<th>$F$ (degrees Fahrenheit)</th>
<th>50</th>
<th>70</th>
<th>90</th>
<th>110</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C$ (degrees Celsius)</td>
<td>10</td>
<td>$21 \frac{1}{9} \left(= \frac{190}{9}\right)$</td>
<td>$32 \frac{2}{9} \left(= \frac{280}{9}\right)$</td>
<td>$43 \frac{1}{9} \left(= \frac{390}{9}\right)$</td>
</tr>
</tbody>
</table>

1. Represent this relationship graphically.

2. What is the ‘constant adder’ in $C$ that corresponds to a 1 unit increase in $F$?

3. Write down an algebraic rule for converting $F$ into $C$.

4. Use your rule to convert
   a. $95^\circ F$ into $^\circ C$
   b. $360^\circ F$ into $^\circ C$

5. Develop an algebraic rule for converting $C$ into $F$.

6. Hence or otherwise convert
   a. $0^\circ C$ into $^\circ F$
   b. $40^\circ C$ into $^\circ F$
6. **Signal and Noise.**

Recall the statement made earlier:

In studies, in laboratories and elsewhere, the idea is to make orderly changes to the *explanatory variable* and observe the variation in the *response variable*.

It is time to look at the insights into change that can be had when this approach is applied to a simple system like the one pictured.

**EAT 3**

Set up an experiment that will allow you to gather information about the way that the length of a helical spring (hung at one end from a fixed point as illustrated) changes when a range of weights are suspended from the other end.

Describe the variables that will be studied in your experiment and the roles that they play.

Predict the results of your experiment.

Conduct the experiment, record the results and compare with your predictions.
6.1 Modelling stochastic change.

1. Complete a table similar to the following with the data that you obtained from your experiment.

<table>
<thead>
<tr>
<th>consecutive adders in $m$</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$w$ (weight in grams)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$l$ (spring length in cm)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>consecutive adders in $l$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For the constant additive change that you affected in the explanatory variable, the change in the response variable was probably not exactly a constant additive one, but sort of ...

What does this suggest about a graphical representation of the relationship between spring length and weight?

2. Represent this relationship graphically.

From your graph, two things should be clear.
- This system is not completely deterministic; there is some unpredictability in the change that we are studying.
- The relationship between the variables is roughly linear.

As a consequence of these realisations, two more things should be apparent
- We can approximate the relationship between $m$ and $l$ with a linear function.
- Such a function will represent the general trend but not the unpredictable change, and so any use of such a model will not take into account this behaviour.

This leaves us with two questions
- what linear function would best “fit” this relationship, and
- how to find it?

4 If you cannot complete the experiment, use the results obtained by the authors,

<table>
<thead>
<tr>
<th>$w$ (weight in grams)</th>
<th>0</th>
<th>50</th>
<th>100</th>
<th>150</th>
<th>200</th>
<th>250</th>
</tr>
</thead>
<tbody>
<tr>
<td>$l$ (spring length in cm)</td>
<td>20</td>
<td>20.4</td>
<td>22</td>
<td>25</td>
<td>27</td>
<td>28</td>
</tr>
</tbody>
</table>
6.2 **Seeking a line of best fit**

1. Using your choice of electronic technology, enter your data into a spreadsheet.

2. In two cells (e.g. D2 and E2), enter an estimate for the gradient/slope and \( l \)-intercept of your linear model for \( l \) in terms of \( w \).

3. Use these values within a formula that will calculate your model’s “predicted” \( l \) values for the \( w \) values used in your experiment (see Column C)\(^5\)

4. Adjust your estimates for the gradient/slope and \( l \)-intercept of your linear model until you think you have found the equation of your “line of best fit”.

5. Compare the equation of your “line of best fit” with that of your classmates.

**But what is the BEST line of best fit?**

- One way to start is to average the measured consecutive adders in \( l \) to estimate your constant adder (i.e. gradient/slope).

- A graph of \( w \) against model \( l \) and actual \( l \) is very helpful.

- Characteristic of a good line of best fit is that, generally, there will be roughly as many points where model \( l \) is greater that actual \( l \) (i.e. above on graph) as there are points where model \( l \) is less than actual \( l \) (i.e. below on graph). These ‘gaps’ should be as small as possible and should be as random as possible in their distribution.

- The BEST line of best fit is one that minimises the sum of the squares of the errors, where the errors are the differences between the model \( l \) and actual \( l \) for each \( w \) value.

- Your spreadsheet can be added to, to measure errors and the sum of their squares.

---

\(^5\) See Section 12.2 for support in using a CASIO ClassPad to build this spreadsheet.
6.3 Using a stochastic model.

Once the line of best fit is determined and you have its equation then you can use it to predict other values that the experiment did not reveal.

Discuss why have the word predict is used in this context.

Interpolation and Extrapolation.

Using your model to predict length values for mass values in between the smallest and largest masses in the experimental data (the “poles” of the data) is called interpolating (“within the poles”).

Using your model to predict length values for mass values outside of the smallest and largest masses in the experimental data is called extrapolating (“beyond the poles”).

The accuracy of an interpolation depends on one feature of the original data.

1. What is this feature??

The accuracy of an extrapolation depends on this same feature of the original data and a rather large assumption in most cases.

2. What is the rather large assumption?

3. Discuss the validity of this assumption in the case of the spring experiment.

7.1 Atmospheric Carbon Dioxide (CO₂) over time – part 1.

For many years the concentration of Carbon Dioxide in the atmosphere (measured in p.p.m. – parts per million) has been recorded at Mauna Loa, Hawaii. The most recent 8 years of data from Mauna Loa are recorded below.

<table>
<thead>
<tr>
<th>Year</th>
<th>2000</th>
<th>2001</th>
<th>2002</th>
<th>2003</th>
<th>2004</th>
<th>2005</th>
<th>2006</th>
<th>2007</th>
</tr>
</thead>
<tbody>
<tr>
<td>CO₂</td>
<td>369.48</td>
<td>371.02</td>
<td>373.1</td>
<td>375.64</td>
<td>377.38</td>
<td>380.99</td>
<td>382.71</td>
<td>384.5</td>
</tr>
</tbody>
</table>

1. Represent this data graphically.
2. By considering the average additive change in $C$, the quantity of atmospheric CO₂ in p.p.m., over successive years, write down a linear model for $C$ in terms of $t$, the number of years since 2000.
3. Sketch your model from part 2 on your graph from part 1.
4. Refine, if necessary, the co-efficients of your linear model in the light on this sketch.
5. Use your linear model to predict the concentration of Carbon Dioxide in the atmosphere in the year 2010.
6. Scientist warn of the potential significant climate change if the concentration of Carbon Dioxide in the atmosphere exceeds 400 p.p.m. According to your model, in what year will that quantity be exceeded?
7. What assumption are you making in your answering on part 6?

7.2 A stochastic process.

A stochastic process has generated the following data:

<table>
<thead>
<tr>
<th>$x$</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
<th>35</th>
<th>40</th>
<th>45</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>997</td>
<td>932</td>
<td>893</td>
<td>825</td>
<td>837</td>
<td>797</td>
<td>727</td>
<td>646</td>
<td>605</td>
<td>611</td>
</tr>
</tbody>
</table>

1. Draw a scatter plot of this data.
2. Calculate the average additive change in $y$ for a one-unit increase in $x$, and hence develop a linear model for $y$ in terms of $x$.
3. Sketch your model from part 2 on your graph from part 1.
4. Refine, if necessary, the co-efficients of your linear model in the light on this sketch.
5. Use your linear model to predict the result of this stochastic process if $x = 65$. 

© A Harradine & A Lupton  Draft, Oct 2008 WIP
7.3 AFL – team success and individual acclaim.

At the end of the 2008 AFL Australian Rules Football season data was gathered on each team including: its ladder position, the number of games won and the total number of Brownlow Medal (fairest and most brilliant player in the AFL) votes awarded to its players. This data was as follows:

<table>
<thead>
<tr>
<th>Team</th>
<th>Ladder Position</th>
<th>No. of games won</th>
<th>Total Brownlow Votes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Geelong</td>
<td>1</td>
<td>21</td>
<td>112</td>
</tr>
<tr>
<td>Hawthorn</td>
<td>2</td>
<td>17</td>
<td>91</td>
</tr>
<tr>
<td>Western Bulldogs</td>
<td>3</td>
<td>15.5</td>
<td>83</td>
</tr>
<tr>
<td>St Kilda</td>
<td>4</td>
<td>13</td>
<td>75</td>
</tr>
<tr>
<td>Adelaide</td>
<td>5</td>
<td>13</td>
<td>78</td>
</tr>
<tr>
<td>Sydney</td>
<td>6</td>
<td>12.5</td>
<td>78</td>
</tr>
<tr>
<td>North Melbourne</td>
<td>7</td>
<td>12.5</td>
<td>58</td>
</tr>
<tr>
<td>Collingwood</td>
<td>8</td>
<td>12</td>
<td>65</td>
</tr>
<tr>
<td>Richmond</td>
<td>9</td>
<td>11.5</td>
<td>69</td>
</tr>
<tr>
<td>Lions</td>
<td>10</td>
<td>10</td>
<td>66</td>
</tr>
<tr>
<td>Carlton</td>
<td>11</td>
<td>10</td>
<td>68</td>
</tr>
<tr>
<td>Essendon</td>
<td>12</td>
<td>8</td>
<td>45</td>
</tr>
<tr>
<td>Port Adelaide</td>
<td>13</td>
<td>7</td>
<td>57</td>
</tr>
<tr>
<td>Fremantle</td>
<td>14</td>
<td>6</td>
<td>56</td>
</tr>
<tr>
<td>West Coast</td>
<td>15</td>
<td>4</td>
<td>31</td>
</tr>
<tr>
<td>Melbourne</td>
<td>16</td>
<td>3</td>
<td>24</td>
</tr>
</tbody>
</table>

1. Represent the relationship between the ladder position and total number of Brownlow Medal votes for the AFL teams in 2008 graphically.
2. Define two variables, and their roles, in relation to this scatter plot.
3. Calculate the average additive change in your response variable for a 1-unit increase in your explanatory variable.
4. Hence develop a linear model of best fit for the relationship between the ladder position and total number of Brownlow Medal votes for the AFL teams in 2008.
5. Express the average additive change from your linear model in a sentence that would be understandable to an AFL supporter.
6. Repeat parts 1 to 5 in relation to the number of games won and total number of Brownlow Medal votes gained.
7. Which of the two explanatory variables used above is more appropriate to explain the variation in the total number of Brownlow Medal votes gained by an AFL team in 2008. Give a reason for your answer.

7.4 The Price of Diamonds.

To investigate exactly how the price of a diamond relates to its weight, a random sample of 40 loose diamonds for sale through an online gem dealer was taken. All were “round cut” and were classified as having clarity “VSI” (very slight inclusions) and colour D to H (near colourless).
The weight \((w\) in carats\) and price \((p\) in $\) of these diamonds is recorded in the following table:

<table>
<thead>
<tr>
<th>(w)</th>
<th>0.27</th>
<th>0.27</th>
<th>0.28</th>
<th>0.25</th>
<th>0.31</th>
<th>0.35</th>
<th>0.36</th>
<th>0.32</th>
<th>0.36</th>
</tr>
</thead>
<tbody>
<tr>
<td>(p)</td>
<td>509</td>
<td>509</td>
<td>518</td>
<td>582</td>
<td>597</td>
<td>660</td>
<td>661</td>
<td>670</td>
<td>670</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(w)</th>
<th>0.3</th>
<th>0.36</th>
<th>0.33</th>
<th>0.28</th>
<th>0.31</th>
<th>0.31</th>
<th>0.33</th>
<th>0.33</th>
<th>0.3</th>
</tr>
</thead>
<tbody>
<tr>
<td>(p)</td>
<td>696</td>
<td>700</td>
<td>727</td>
<td>747</td>
<td>782</td>
<td>782</td>
<td>800</td>
<td>808</td>
<td>829</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(w)</th>
<th>0.37</th>
<th>0.32</th>
<th>0.32</th>
<th>0.39</th>
<th>0.39</th>
<th>0.41</th>
<th>0.4</th>
<th>0.4</th>
<th>0.39</th>
</tr>
</thead>
<tbody>
<tr>
<td>(p)</td>
<td>851</td>
<td>852</td>
<td>859</td>
<td>890</td>
<td>890</td>
<td>981</td>
<td>1017</td>
<td>1068</td>
<td>1113</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(w)</th>
<th>0.39</th>
<th>0.42</th>
<th>0.42</th>
<th>0.42</th>
<th>0.43</th>
<th>0.42</th>
<th>0.46</th>
<th>0.46</th>
<th>0.48</th>
</tr>
</thead>
<tbody>
<tr>
<td>(p)</td>
<td>1113</td>
<td>1119</td>
<td>1310</td>
<td>1331</td>
<td>1398</td>
<td>1476</td>
<td>1516</td>
<td>1543</td>
<td>1595</td>
</tr>
</tbody>
</table>

1. Draw a scatter plot of \(p\) against \(w\).
2. Draw on this scatter plot the line of best fit (determined ‘by eye’).
3. By obtaining accurate information from the line of best fit that you have drawn, determine the equation of your line of best fit.
4. Provide an interpretation for the slope/gradient of your equation.
5. Use your model to predict
   a. The cost of a diamond weighing 0.45 carats
   b. The cost of a diamond weighting 0.55 carats.
   c. The size of diamond likely to cost $2000.

7.5 Unleaded Petroleum and the price of Crude Oil.

An economics teacher mentions to their class that a one dollar increase in the barrel price of Crude Oil adds one cent per litre to the price of unleaded petrol.

To investigate the accuracy of this “rule of thumb” the following data has been collected:

### 2006

<table>
<thead>
<tr>
<th>Month</th>
<th>Jan</th>
<th>Feb</th>
<th>Mar</th>
<th>Apr</th>
<th>May</th>
<th>Jun</th>
<th>Jul</th>
<th>Aug</th>
<th>Sep</th>
<th>Oct</th>
<th>Nov</th>
<th>Dec</th>
</tr>
</thead>
<tbody>
<tr>
<td>Crude Oil ($US/barrel)</td>
<td>58.3</td>
<td>54.65</td>
<td>55.42</td>
<td>62.5</td>
<td>62.94</td>
<td>62.85</td>
<td>66.28</td>
<td>64.93</td>
<td>55.73</td>
<td>50.98</td>
<td>50.98</td>
<td>54.06</td>
</tr>
<tr>
<td>ULP (cents/litre)</td>
<td>117.7</td>
<td>117.9</td>
<td>120.6</td>
<td>128.6</td>
<td>135.6</td>
<td>135.5</td>
<td>135</td>
<td>133.3</td>
<td>122.3</td>
<td>116.3</td>
<td>112.7</td>
<td>115.7</td>
</tr>
</tbody>
</table>

### 2007

<table>
<thead>
<tr>
<th>Month</th>
<th>Jan</th>
<th>Feb</th>
<th>Mar</th>
<th>Apr</th>
<th>May</th>
<th>Jun</th>
<th>Jul</th>
<th>Aug</th>
<th>Sep</th>
<th>Oct</th>
<th>Nov</th>
<th>Dec</th>
</tr>
</thead>
<tbody>
<tr>
<td>Crude Oil ($US/barrel)</td>
<td>46.53</td>
<td>51.36</td>
<td>52.64</td>
<td>56.08</td>
<td>55.43</td>
<td>59.25</td>
<td>65.96</td>
<td>64.23</td>
<td>70.94</td>
<td>77.56</td>
<td>86.92</td>
<td>83.46</td>
</tr>
<tr>
<td>ULP (cents/litre)</td>
<td>115.2</td>
<td>114.9</td>
<td>122.5</td>
<td>124.8</td>
<td>130.2</td>
<td>129.9</td>
<td>126.7</td>
<td>122.7</td>
<td>123.8</td>
<td>124.3</td>
<td>130.5</td>
<td>137.1</td>
</tr>
</tbody>
</table>
1. Define the variables and their roles in this situation.
2. Develop a model that best fits this data.
3. With reference to your model, comment on the “rule of thumb” quoted above. If necessary, formulate a revised rule of thumb.
4. Use your model to predict
   a. The price of ULP if crude oil cost $80 per barrel.
   b. The level to which the cost of crude oil will have risen if the cost of ULP reaches $8.00 per litre (CSIRO worst case scenario for 2018).
8. Invest smarter.

Compound Interest … explains why $1000 … can grow to $47 000 over 50 years. No wonder Albert Einstein described compound interest as one of the greatest human discoveries

EAT 4 How does Compound Interest work?

Consider an investment of $1,000 that earns 6% interest p.a.
In the 1st year, how much interest is earned?
Therefore, what is the total value of the investment after the 1st year?

Based on this,
In the 2nd year, how much interest is earned?
Therefore, what is the value of the investment after the 2nd year?

Repeat this for the 3rd, 4th and 5th years.

Can you write down a model for $V$, the value of the investment after $n$ years?
Is this a deterministic or stochastic model? Why?

If you can write down a model for $V$ in terms of $n$, use it to predict the value of the investment after 50 years.
If this value is not $47,000, how would the scenario have to changed if the claim above was to be accurate?

What different ways were used to perform the calculations required in EAT 4?
Which way was easiest?
How do you (easily) increase an amount by 15%? 80%? 1%?
Building a model for compound interest.

In **EAT 4** you should have performed the equivalent to the following calculations

<table>
<thead>
<tr>
<th>n</th>
<th>V (Value in $ at end of n th year)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1000</td>
</tr>
<tr>
<td>1</td>
<td>1000 × 1.06 = 1060</td>
</tr>
<tr>
<td>2</td>
<td>1060 × 1.06 = 1123.60</td>
</tr>
<tr>
<td>3</td>
<td>1123.60 × 1.06 = 1191.02</td>
</tr>
<tr>
<td>4</td>
<td>1191.02 × 1.06 = 1262.48</td>
</tr>
<tr>
<td>5</td>
<td>1262.48 × 1.06 = 1338.23</td>
</tr>
</tbody>
</table>

This process of repeated multiplication by 1.06 can be examined further

<table>
<thead>
<tr>
<th>n</th>
<th>V</th>
<th>V</th>
<th>V</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1000</td>
<td>1000</td>
<td>1000×1.06 0</td>
</tr>
<tr>
<td>1</td>
<td>1000×1.06 = 1060</td>
<td>(1000)×1.06</td>
<td>1000×1.06 1</td>
</tr>
<tr>
<td>2</td>
<td>1060×1.06 = 1123.60</td>
<td>(1000×1.06)×1.06</td>
<td>1000×1.06 2</td>
</tr>
<tr>
<td>3</td>
<td>1123.60×1.06 = 1191.02</td>
<td>(1000×1.06×1.06)×1.06</td>
<td>1000×1.06 3</td>
</tr>
<tr>
<td>4</td>
<td>1191.02×1.06 = 1262.48</td>
<td>(1000×1.06×1.06×1.06)×1.06</td>
<td>1000×1.06 4</td>
</tr>
<tr>
<td>5</td>
<td>1262.48×1.06 = 1338.23</td>
<td>(1000×1.06×1.06×1.06×1.06)×1.06</td>
<td>1000×1.06 5</td>
</tr>
</tbody>
</table>

From this table it should be clear that the deterministic model for \( V \) in terms of \( n \) is

\[
V = 1000(1.06)^n
\]

This is an example of a **simple exponential function**

\[
y = a \times b^x
\]

This has a direct parallel with the linear function \( y = mx + c \) where \( c \) and \( m \) represent the \( y \)-intercept and the constant adder respectively.

### 8.1 Can you use the knowledge?

1. Explain why a constant multiplier of 1.06 implies that the previous value is increased by 6%.

2. What sort of change would be created by a constant multiplier of
   a. 1.45?
   b. 1.025?
   c. 0.95?
   d. 0.825?

3. Write down a function that would model \( V \), the value of an investment of
   a. $6250 invested at 8.5% compound interest for \( n \) years.
   b. $14.92 invested at 2.25% compound interest for \( n \) years.
   c. $115 000 invested at 11% compound interest for \( n \) years.
9. Simple exponential functions.

**EAT 5 Investigating the graph of** \( y = a \times b^x \).

Select a fixed value for \( a \).
Choose a range of values for \( b \) (what sort of numbers could you use?)
Using your \( a \) value, and a different \( b \) value for each, draw 4 or 5 sketches of \( y = a \times b^x \) on the same large set of axes.
Summarise your findings with respect to the effect that different \( b \) values have on the graph of \( y = a \times b^x \).
Repeat this investigation with a fixed \( b \) value and a range of values for \( a \).

Glass - thickness and transparency.

A glass manufacturer plans to produce a type of glass where its transparency (the amount of light that passes through it) depends on its thickness. According to glass making theory the relationship will be as follows,

<table>
<thead>
<tr>
<th>Glass Thickness ( x ) (mm)</th>
<th>+1</th>
<th>+1</th>
<th>+1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Transparency ( T ) (lumens)</td>
<td>100</td>
<td>80</td>
<td>64</td>
</tr>
</tbody>
</table>

Like compound interest calculations, this deterministic relationship between glass thickness and transparency exhibits the following property,

**For a constant additive change in the explanatory variable, a constant multiplicative change occurs in the response variable.**

This is sometimes referred to as exponential or multiplicative change.
In such a case, where a constant multiplier (i.e. 0.8) is evident and an initial value is provided, a simple exponential model is appropriate and easy to obtain i.e.

\[
T = 100 \times 0.8^x
\]

“Based on the data below, predict the \( x \) value when \( y = 900 \)"

<table>
<thead>
<tr>
<th>( x )</th>
<th>3</th>
<th>6</th>
<th>9</th>
<th>12</th>
<th>15</th>
<th>18</th>
<th>21</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>7</td>
<td>12</td>
<td>21</td>
<td>36</td>
<td>62</td>
<td>107</td>
<td>184</td>
</tr>
</tbody>
</table>

**Finding a simple exponential model.**

Determining the average multiplicative change in the \( y \) values above

\[(1.714 + 1.75 + 1.714 + 1.722 + 1.726 + 1.720) / 6 = 1.724\]

We know that, if \( y = a \times b^x \) then \( b \times b \times b = b^3 = 1.724 \)

As the multiplicative change of 1.724 is caused by a 3-unit increase in \( x \)

(i.e. by three applications of the constant multiplier).

Hence \( b = \sqrt[3]{1.724} = 1.20 \).

Further \( a = 7 \div 1.20 \div 1.20 \div 1.20 = 7 \div 1.724 = 4.06 \)

This is working backwards to the “initial value” – the value of \( y \) when \( x = 0 \).

So our model for \( y \) in terms of \( x \) is \( y = 4.06 \times 1.2^x \)

**Using a simple exponential model.**

Our question can now be answered by solving \( 4.06 \times 1.2^x = 900 \), ... but how ...?

**Method 1 – Tabular**

A rough answer

```plaintext
10.1 Seismic Energy – the Richter Scale
```

The Richter scale represents the amount of seismic energy released during an earthquake or other event. Richter magnitudes can be equated to other units of energy like mega joules (MJ).

The table below provides the mega joule equivalent to six Richter magnitudes

<table>
<thead>
<tr>
<th>consecutive adders in ( R )</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( R ) (Richter magnitude)</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>( M ) (energy in megajoules)</td>
<td>4.2</td>
<td>132.8</td>
<td>4200</td>
<td>132816</td>
<td>4200000</td>
<td>132815662</td>
</tr>
</tbody>
</table>

**Method 2 – Graphical**

Sketch the model.

Get a View Window containing the solution.

Perform an \( x \)-Cal

**Analysis:** \( G \text{-Solv} \)

then \( X \text{-Cal} \).

**Method 3 – Algebraic**

\[4.06 \times 1.2^x = 900\]

\[\therefore 1.2^x = 221.675\]

\[\therefore x = \log_{1.2} 221.675\]

\[\therefore x = 29.625\]

(calc’s to 3 dec. places)

10.1 Seismic Energy – the Richter Scale

The Richter scale represents the amount of seismic energy released during an earthquake or other event. Richter magnitudes can be equated to other units of energy like mega joules (MJ).

The table below provides the mega joule equivalent to six Richter magnitudes

<table>
<thead>
<tr>
<th>consecutive adders in ( R )</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( R ) (Richter magnitude)</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>( M ) (energy in megajoules)</td>
<td>4.2</td>
<td>132.8</td>
<td>4200</td>
<td>132816</td>
<td>4200000</td>
<td>132815662</td>
</tr>
</tbody>
</table>
1. Using the previous table, write down a model for \( M \) in terms of \( R \).

2. Use this model to determine the energy released (in MJ) by the earthquake that devastated Kashmir in 2005 which measured 7.5 on the Richter Scale.

3. The Valdivia Earthquake that struck Chile in 1960 measured 9.5 on the Richter Scale, one of the highest ever recorded. How many MJ of energy did it release?

4. Describe the difference in energy release between earthquakes that differ by two on the Richter scale.

### 10.2 Depreciation.

According to a company’s Depreciation Schedule, an asset’s “book value” in the years after purchase is as follows:

<table>
<thead>
<tr>
<th>( t ) (years since purchase)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V ) (book value – whole $’s)</td>
<td>112500</td>
<td>95625</td>
<td>81281</td>
<td>69089</td>
<td>58725</td>
<td>49916</td>
</tr>
</tbody>
</table>

1. Interpret the constant multiplier evident in the relationship between \( V \) and \( t \).

2. Write down a simple exponential model for this relationship.

3. Hence determine the book value of the asset
   a. 10 years after purchase.
   b. 13.5 years after purchase.

4. In what year does the book value of the asset fall below $5000?

### 10.3 Radioactive Decay.

The radioactive isotope Thorium 234 decays (by beta-emission of radioactivity into protactinium-234) in the following way:

<table>
<thead>
<tr>
<th>( t ) (days)</th>
<th>0</th>
<th>4</th>
<th>8</th>
<th>12</th>
<th>16</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>( W ) (mass of remaining Thorium-234 in grams)</td>
<td>100</td>
<td>89</td>
<td>79.4</td>
<td>70.7</td>
<td>63</td>
<td>56.1</td>
</tr>
</tbody>
</table>

1. Determine the constant multiplier in \( W \) for a one-unit increase in \( t \).
   Interpret this value in the context of the question.

2. Write down a simple exponential model for this relationship.

3. Hence the amount of Thorium-234 remaining after
   a. 7 days.
   b. 365 days.

4. What is the half-life of Thorium-234?
   (An isotope’s half-life is the time taken for it to decay to half its mass)
10.4 Sound Intensity and Loudness.

Sound intensity, a measure of the energy travelling in sound waves, is measured in watts per m$^2$. Sound loudness is measured in decibels. The two are related in the following way:

<table>
<thead>
<tr>
<th>$d$ (db - decibels)</th>
<th>0</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I$ (watts per m$^2$)</td>
<td>$10^{-12}$</td>
<td>$10^{-11}$</td>
<td>$10^{-10}$</td>
<td>$10^{-9}$</td>
<td>$10^{-8}$</td>
<td>$10^{-7}$</td>
</tr>
</tbody>
</table>

1. Determine the constant multiplier in $I$ for a one-unit increase in $d$.
2. Write down a simple exponential model for this relationship.
3. Hence determine the Intensity, in watts per m$^2$, of loud with loudness of:
   a. 56 db.
   b. 105 db.

10.5 More on compound interest.

Investments and loans are commonly compounded more often than annually. If so then the quoted interest rate is divided by the number of compounds per year, and the (new and smaller) constant multiplier is “applied” once per compounding period. For example, if our $1000 was to earn 6% compounded quarterly (4 times per year) then our model for its value would become

$$V = 1000(1.015)^n$$ where $n$ is now the length of the investment in quarters.

1. Determine the value of our $1000 investment after 50 years of earning 6% compounded quarterly.
2. Write down a model for $V$ if the investment were compounded monthly.
3. Hence, determine its value after 50 years in this situation.
4. Repeat part 2 and 3 with interest that is compounded daily.

Taking compounding to the limit ... 

5. Consider $1 invested at 100% p.a. (a theoretical investment obviously).
   Calculate the value of this “investment” after 1 year if the interest was compounded with increasing frequency i.e.

<table>
<thead>
<tr>
<th>Compounding period</th>
<th>quarterly</th>
<th>monthly</th>
<th>daily</th>
<th>every second</th>
<th>continuously</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value of investment</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

11.1 Managing a Gas Field – 2

Revisit the Managing a Gas Field scenario on page 5. Recall your prediction about the month in which the 6th gas well was to be required. Now we can use our increased knowledge of algebraic models to refine this prediction.

1. If you have not already, draw a scatter plot of the gas flow data on page 5.
2. What property of the data (or of the context of gas extraction) suggests that a simple exponential function could model the relationship between flow and time?
3. Investigate an approximate constant multiplier in the change in flow.
4. Hence formulate an algebraic model for the relationship between flow and time.
5. Use your model to predict when the 6th well will be required.
6. Compare this with your earlier prediction. How did you do?

As it happens, more data is available from this gas field as follows

<table>
<thead>
<tr>
<th>Month (end date)</th>
<th>Relative time (months)</th>
<th>Rate of Gas Flow f (MMscf/d)</th>
</tr>
</thead>
<tbody>
<tr>
<td>6/30/1999</td>
<td></td>
<td>12.992</td>
</tr>
<tr>
<td>7/31/1999</td>
<td></td>
<td>9.21</td>
</tr>
<tr>
<td>8/31/1999</td>
<td></td>
<td>8.836</td>
</tr>
<tr>
<td>9/30/1999</td>
<td></td>
<td>5.874</td>
</tr>
<tr>
<td>10/31/1999</td>
<td></td>
<td>4.938</td>
</tr>
<tr>
<td>11/30/1999</td>
<td></td>
<td>11.775</td>
</tr>
<tr>
<td>12/31/1999</td>
<td></td>
<td>16.709</td>
</tr>
<tr>
<td>1/31/2000</td>
<td></td>
<td>15.579</td>
</tr>
<tr>
<td>2/29/2000</td>
<td></td>
<td>14.861</td>
</tr>
<tr>
<td>3/31/2000</td>
<td></td>
<td>14.067</td>
</tr>
<tr>
<td>5/31/2000</td>
<td></td>
<td>28.882</td>
</tr>
<tr>
<td>6/30/2000</td>
<td></td>
<td>24.963</td>
</tr>
<tr>
<td>7/31/2000</td>
<td></td>
<td>23.124</td>
</tr>
<tr>
<td>8/31/2000</td>
<td></td>
<td>20.43</td>
</tr>
<tr>
<td>9/30/2000</td>
<td></td>
<td>18.963</td>
</tr>
<tr>
<td>10/31/2000</td>
<td></td>
<td>17.335</td>
</tr>
<tr>
<td>11/30/2000</td>
<td></td>
<td>15.61</td>
</tr>
<tr>
<td>12/31/2000</td>
<td></td>
<td>14.516</td>
</tr>
</tbody>
</table>

7. In what month was the 6th well required? How good was your prediction?
8. Does it seem as if the 6th well was ready at that time?
9. Using the additional data, develop a prediction for when the flow of gas from this field will next fall below 5 MMscf/d, making the field unviable.
11.2 Bouncing ball

A scientist dropped a steel ball from a height of 10 metres onto a very hard surface, and allowed it to bounce several times, yielding the following data:

<table>
<thead>
<tr>
<th>Bounce number (n)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
</tr>
</thead>
<tbody>
<tr>
<td>Height after bounce (H)</td>
<td>10</td>
<td>8.6</td>
<td>7.4</td>
<td>6.0</td>
<td>5.1</td>
<td>4.5</td>
<td>4.0</td>
<td>3.2</td>
<td>2.8</td>
<td>2.2</td>
<td>2.0</td>
<td>1.8</td>
<td>1.5</td>
<td>1.2</td>
<td>1.0</td>
</tr>
<tr>
<td>Consecutive adders in n</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Consecutive multipliers in H</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

1. Fill in the last two rows of this table.
2. Draw a scatter plot of H against n.
3. Use the values in the table to write down a simple exponent model for H in terms of n.
4. Represent this model on your scatter plot.
5. Explain why, in this context, interpolation is of little relevance.
6. Using your model, predict the height of the 20th bounce.
7. Using your model, predict which will be the first bounce with a height of less than 10 centimetres.

11.3 Another stochastic process.

A stochastic process has generated the following data:

<table>
<thead>
<tr>
<th>x</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>103.2</td>
<td>101.9</td>
<td>105.1</td>
<td>111.4</td>
<td>111.7</td>
<td>118.5</td>
<td>115.4</td>
<td>121.6</td>
<td>126.7</td>
</tr>
</tbody>
</table>

1. Draw a scatter plot of this data.
2. Calculate the average multiplicative change in y for a one-unit increase in x.
3. Use your answer to part 2 to develop an approximate value for y when x = 0, and hence develop a simple exponential model for y in terms of x.
4. Sketch your model from part 3 on your graph from part 1.
5. Refine, if necessary, the co-efficients of your simple exponential model in the light on this sketch.
6. Use your linear model to predict the result of this stochastic process if x = 25.
11.4 Worldwide use of the WWW.

According to an internet monitor, the percentage of the world’s population that uses
the internet (at least monthly) has changed, since 1995, in the following way:

<table>
<thead>
<tr>
<th>years since '95</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>WWW use (%)</td>
<td>0.4</td>
<td>0.9</td>
<td>1.7</td>
<td>3.6</td>
<td>4.1</td>
<td>5.8</td>
<td>8.6</td>
<td>9.4</td>
<td>11.1</td>
<td>12.7</td>
<td>15.7</td>
<td>16.7</td>
<td>20</td>
</tr>
</tbody>
</table>

1. Represent this data on a scatter plot.
2. Explain why a linear model is not able to capture the change in use of the
   world wide web since 1995.
3. Investigate the possibility of modelling this change with a simple exponential
   model.
4. Use technology to obtain a quadratic model of best fit for this data.
5. Use your preferred model from parts 3 or 4 to predict when a third of the world
   will use the internet (at least monthly).
6. How accurate do you think the prediction asked for in part 5 is likely to be?

11.5 Global Temperature and Atmospheric CO₂.

The table provided contains the annual data from Mauna Loa, the longest running
atmospheric monitoring station based in Hawaii, along with the global mean land-
ocean temperature (relative to the 1951-1980 mean) for the years after 1959.

Investigate appropriate models for
1. atmospheric CO₂ in terms of years since 1959,
2. global mean land-ocean temperature in terms of years since 1959
3. global mean land-ocean temperature in terms of atmospheric CO₂.

In each case,

a. Assign variables and classify their role.

b. Determine the type of algebraic model that you will use to describe the
   relationship between the variables, explaining the reasons for your selection.

c. Determine the equation of the line / curve of best fit, using the concepts of
   “constant adder / multiplier” and a “graph and refine” method, with
   documentation.

d. Use the model fitting capacity of your choice of technology to find the
   equation of an alternative line / curve of best fit.

e. Compare the models obtained in parts c and d and select your choice of
   model. Give reasons for your selection.

f. Use your choice of model to make predictions about 2020.
g. Use technology to obtain a quadratic model of best fit (if appropriate).

h. Compare this model with your choice in part e in terms of
i. fit to data
ii. predictions about 2020.

i. Discuss the limitations inherent in your results.

<table>
<thead>
<tr>
<th>year</th>
<th>Global atmospheric concentrations of carbon dioxide</th>
<th>global mean land-ocean temperature (relative to 1951-1980 mean)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1959</td>
<td>315.83</td>
<td>0.06</td>
</tr>
<tr>
<td>1960</td>
<td>316.75</td>
<td>-0.01</td>
</tr>
<tr>
<td>1961</td>
<td>317.49</td>
<td>0.08</td>
</tr>
<tr>
<td>1962</td>
<td>318.3</td>
<td>0.04</td>
</tr>
<tr>
<td>1963</td>
<td>318.83</td>
<td>0.08</td>
</tr>
<tr>
<td>1965</td>
<td>319.87</td>
<td>-0.11</td>
</tr>
<tr>
<td>1966</td>
<td>321.21</td>
<td>-0.03</td>
</tr>
<tr>
<td>1967</td>
<td>322.02</td>
<td>0</td>
</tr>
<tr>
<td>1968</td>
<td>322.89</td>
<td>-0.04</td>
</tr>
<tr>
<td>1969</td>
<td>324.46</td>
<td>0.08</td>
</tr>
<tr>
<td>1970</td>
<td>325.52</td>
<td>0.03</td>
</tr>
<tr>
<td>1971</td>
<td>326.16</td>
<td>-0.1</td>
</tr>
<tr>
<td>1972</td>
<td>327.29</td>
<td>0</td>
</tr>
<tr>
<td>1973</td>
<td>329.51</td>
<td>0.14</td>
</tr>
<tr>
<td>1974</td>
<td>330.08</td>
<td>-0.08</td>
</tr>
<tr>
<td>1975</td>
<td>330.99</td>
<td>-0.05</td>
</tr>
<tr>
<td>1976</td>
<td>331.98</td>
<td>-0.16</td>
</tr>
<tr>
<td>1977</td>
<td>333.73</td>
<td>0.13</td>
</tr>
<tr>
<td>1978</td>
<td>335.34</td>
<td>0.02</td>
</tr>
<tr>
<td>1979</td>
<td>336.68</td>
<td>0.09</td>
</tr>
<tr>
<td>1980</td>
<td>338.52</td>
<td>0.18</td>
</tr>
<tr>
<td>1981</td>
<td>339.76</td>
<td>0.27</td>
</tr>
<tr>
<td>1982</td>
<td>340.96</td>
<td>0.05</td>
</tr>
<tr>
<td>1983</td>
<td>342.61</td>
<td>0.26</td>
</tr>
</tbody>
</table>

NB: A good way to demonstrate the skills that you have learned in this unit would be to complete Section 11.5 in the form of a mathematical report. To do this:

- Link the subsections with sentences or paragraphs that show that you know what you are doing and why you are doing it as well as what your results mean (this should mean you will not need to refer to part a, b etc).
- Look for ways to extend your work beyond the minimum requirements.
12. eTech Support.

12.1 Representing bi-variate data – the Scatter Plot.

Using the spreadsheet mode in your CASIO ClassPad:
Enter the data into two columns of your spreadsheet.
Select the cells (tap and swipe).
Tap the dropdown “graph selection” arrow
Tap the scatter plot icon (circled)

You should get:

Using in your CASIO ClassPad:
Enter the data set into two Lists.
If you name them – by entering a name into the heading row – then the Lists are stored as variables.

Now tap or SetGraph: Setting...
Enter the settings as shown.

Make sure that this graph is set as Draw:On and that all others are set as Draw:Off.

If you have named them, your XList and YList must be referred to by name.

Tap Set when these settings are correct.

Now tap
12.2 Model fitting via a spreadsheet.

Enter the spreadsheet mode of your CASIO ClassPad.

Text can be used to label aspects of a spreadsheet.
To do so, tap on the cell and enter text using the keyboard.
In this case the labels in Row 2 relate to the slope \( m \) and y-intercept \( c \) that will be later stored in Row 3.

The numerical data can be entered in a similar way (into Column A and Column B say).

Graphing Spreadsheet data.

To graph data that is in a spreadsheet, first select the data, then choose a type of graph from the selection obtained by tapping the drop-down arrow next to the graph icon (circled top right).

Tap \( \text{Plot} \) to obtain a Scatter Plot of the selected data.

Investigating constant adders.

The consecutive adders in the 1-values can be found by entering the formula 
\[ =B6-B5 \]
in Cell C6. This formula can be filled down Column C by tapping \( \text{Edit} \) \( \text{: Fill Range} \) (with cell C6 selected) and setting the Range as C6:C10.
(The \( \text{:} \) is available from top left of the screen as circled below)
The mean ("average") consecutive adder can be calculated by entering the formula
\[ \text{mean(C6:C10)} \] (into Cell C12 say).

Based on this, a 'first guess' for the slope of the linear model would be
\[ m = \frac{16}{30} = 0.032 \]

By combining this with the initial \( l \)-value of 20 as a 'first guess' for the vertical axis intercept, we are ready to seek our line of best fit by storing these values in cells A3 and B3 respectively.

**Seeking a line of best fit.**

Because, in this spreadsheet \( m=A3 \), \( w=A5 \ldots A10 \) and \( c=B3 \), the equation of the line of best fit,
\[ l = m \times w + c \]
becomes the formula \( =A3*A5+B3 \), to be entered into C5 and then filled down using Edit : Fill Range.

**Note:** the $ is necessary so that, when the formula is filled down, the reference to the values in row 3 does not 'move down' to row 4, 5, ... (this 'moving down' is desirable for the \( w \) values in Column A but is not desirable for the \( m \) and \( c \) values).

We can evaluate the degree of 'fit' of this model in a number of ways.

We could graph both the measured \( l \) values and the model's \( l \)-values against \( w \).

This can be made clearer by 'zooming in' by selecting Zoom Box from the drop down menu (next to the Arrow icon circled top right) and then drawing a box over the relevant part of the graph.

This graph more clearly shows that the model (\( x \)'s) 'over-predict' \( l \) for small \( w \) values and under-predict \( l \) for larger values of \( w \).
Measuring Error

Another way to evaluate the degree of ‘fit’ of our model is to compare the predicted and measured l-values numerically. This can be done by ‘eye-balling’ the values in columns B and C.

A way to quantify this ‘fit’ is to measure the ‘error’, that is, the difference between the measured values and the predicted values i.e. measured l – predicted l. In our spreadsheet that means using the formula = B5 - C5 and filling it down over D5:D10.

This process can be seen below.

Now that it is built this, spreadsheet allows you to modify the m and c-values of the linear model in order to improve its fit. Below you can see how the initial values compare to a pair of adjusted m and c-values.
Minimising the sum of squared errors.

One common way of decided the BEST line of best fit is by determining which line has the least sum of squared errors. The spreadsheet can be added to, to make this determination easy.

This addition allows us to confirm that the adjusted values for \( m \) and \( c \) (the second model considered on the previous page) provide a better fitting model that the initial values.

Further refinements to \( m \) and \( c \) may provide a better fitting model with a lower sum of squared errors.
12.3 Working with data and models.

Calculating Additive Change
See Section 12.2, on the use of the ClassPad’s spreadsheet.

Exporting data from a spreadsheet to LIST.
Select a range of cells in the spreadsheet and tap File: Export.
Name the data as a LIST variable.
Tap OK.

Working with user-defined functional models.
Draw a scatter plot of the data in

Tap ![scatter plot icon] and enter the possible functional model into the Graph Editor window.

Now tap ![equation editor icon]

By tapping back into the Graph Editor window, altering the equation of the functional model and re-drawing (tapping ![equation editor icon] again), the functional model can be refined and its “fit” improved.
Calculating with functional models.

Once a satisfactory functional model has been obtained, it can be used to calculate the value of extrapolations and interpolations. This can be done in many ways, one of which is by using $W$.

With a function entered and selected in the Graph Editor window it can be drawn on the current View-Window by tapping $\mathbb{A}$.

**Note 1:** A function is selected if its box is ticked – tap to tick or untick a box.

**Note 2:** The View-Window does not change when moving from $I$ to $W$. This is quite handy when a Scatter Plot of data is drawn prior to a model for the same data.

To calculate $x$ and $y$ values for a functional model the ClassPad’s G-Solve menu is very useful.

Tap Analysis : G-Solve and then either $Y-Cal$ or $X-Cal$ to calculate a $y$ or an $x$ value.

**Note:** if the desired point is not in your View-Window you will be told it is Not Found, whereupon you will need to broaden your View-Window to use this method.
12.4 Obtaining a calculator-defined functional model.

With data sets where it is difficult to determine a value for average additive change, it is sometimes desirable to use a ClassPad to obtain a functional model of best fit.

To do this, first draw a Scatter Plot of the data and then tap CALC.

This presents a range of the types of functional models that can be obtained by the ClassPad.

To obtain the equation of a linear model of best fit, tap Linear Reg.

The Set Calculation window that appears asks for the location / name of our Xlist and YList, as well as giving us the option of Copy Formula (in y1 for example) in the Graph Editor.

The Stat Calculation window (below) shows the co-efficients of the linear function of best fit – calculated by the ClassPad by minimising the sum of squared errors (the value of the minimum sum of squared errors is also provided as MSe).

When you tap OK, the graph of this function is drawn upon the Scatter Plot, to help you to assess its suitability.
12.5 Investigating a family of functions.

The graphs of a number of members of a family of functions like $y = 5 \times b^x$ can be done in W. Enter their equations and tap $\mathbf{\text{+}}$.

The appearance of such a family of functions depend s a lot on your View-Window.

Using the Default View-Window (tap $\mathbf{\text{55}}$ and Default) the graph is as shown.

To improve this representation you can change your View Window.

In the case things are improved by simply tapping $\mathbf{\text{66}}$.

The family of functions $y = a \times b^x$ can be investigated in a more dynamic way.

This is done by first going to $\mathbf{\text{66}}$ and assigning values to $a$ and to $b$ as shown right.

The arrow you need is on the $\mathbf{\text{mth}}$ keyboard.

The $\mathbf{\text{abc}}$ keyboard is needed for the $a$ and the $b$.

Now go to $\mathbf{\text{66}}$ and enter the function as $y = a \times b^x$.

Now tap $\mathbf{\text{66}}$ and Dynamic Graph.
In the **Dynamic Graph** window, enter the values as shown, make sure **Modify** is set to **Manual** and then tap **OK**.

You should see graph something like this one.

To see the effect on the graph of changing the “a” parameter, use and on the Arrow Pad.

To see the effect on the graph of changing the “b” parameter, use and on the Arrow Pad.
13. Answers.

4.1 Can you ....1.

1. \( A = 180n - 360 \)

2. a. 180
   b. 180

   This value represents the increase in internal angle sum upon the addition of a side to a polygon.

3. An axis intercept of \(-360\) means that, according to this relationship, a polygon with no sides would have an internal angle sum of \(-360^\circ\) (a somewhat “un-real” result).

4. \( A = 180 \times 12 - 360 = 1800^\circ \)

5. \( 1680 = 180n - 360 \)
   \[ \Rightarrow n = 11.33... \]
   But \( n \in \mathbb{Z} \) (as polygons only have whole numbers of sides).

6. Rely of the fact that the sum of the interior angles of a triangle is \( 180^\circ \).

5.2

1. A constant adder in \( K \) (i.e. 36) for a constant adder in \( M \) (i.e. 10).

2. \( K = 3.6M \).

3. a. \( 66.96 \) km/h
   b. \( 4320 \) km/h
   c. \( 16 \frac{2}{3} \) m/s
   d. \( 43 \frac{3}{18} \) m/s

5.3

1. This shows the 4 points

   This better captures the relationship.

5. A \( = 216t + 5400 \)

6. a. \( 216 \times 10 + 5400 = 7560 \)
   b. \( 7074 \)

7. \( 216t + 5400 = 9000 \)
   \[ \Rightarrow t = 16 \frac{2}{3} \]
   (i.e. 16 years and 8 months).
7.1
1.

2. \[ C = 2.15t + 369.48 \]
3.

4. \[ C = 2.3t + 368.5 \]

5. \[ \text{391.5 p.p.m.} \]
6. \[ 2013 \]
7. That the current trend continues.

7.2
1.

2. Ave. additive change
   \[ = -42.88 \div 5 \]
   \[ = -8.578 \]
   \[ \therefore y = -8.578x + 1039.889 \]
3.

4. \[ y = -9.3x + 1045 \]
5. \[ y = 440.5 \]

7.3
1.

2. \[ I \] – an AFL team’s ladder position – is the explanatory variable.

3. \[ v = -5.866... \]
4. \[ v = -5.866I + 117.9 \]

5. For every position lower on the ladder a team finished, their player’s received approximately 5.2 fewer Brownlow medal votes, on average.

6. (2). \[ w \] – the number of wins by an AFL team – is the explanatory variable.

   \[ v \] – the number of Brownlow medal votes earned (in total) by players in an AFL team – is the response variable.

6. (3). \[
(5.25 + 5.33 + 2.6 + 17 + 6 - 6 + 0.66 + 11.5 + 1 + 12.5 + 7) \div 11
= 5.71
\]
6. (4). \[ v = 5.71W + 6.87 \]

6. (5) For every additional game an AFL team won, its players received around 5.2 more Brownlow medal votes on average.

7. \[ w \] is a more appropriate explanation of variations in \( v \) as:
   (1) \( I \) is also explained by \( w \).
   (2) the graph of \( w \) against \( v \) shows less non-systematic variation than the graph of \( w \) against \( I \).
7.4

1.

2.

3. \( p = 5000w - 820 \)

4. A diamond weighing 1 carat more will cost $5000 extra.

5.
   a. $1430
   b. $1930
   c. 0.564 carats.

7.5

1. 
   \( c \) - the cost of a barrel of crude oil in US dollars is the explanatory variable.
   
   \( p \) - the price of a litre of ULP in Australian cents is the response variable.

2. 
   \( p = 0.465c + 96 \)

3. The “rule of thumb” suggests that the model should have an adder of approx. 1. The model has an adder of around 0.5. As such, a much better rule of thumb would be that a one dollar increase in the barrel price of crude oil adds half a cent per litre to the price of ULP.

8.1

1. To increase \( x \) by 6% is to add 6% of \( x \) to \( x \) i.e.
   
   \[ x + x \times 6 \div 100 = x(1 + \frac{6}{100}) = x \times 1.06 \]

2. a. Increase by 45% 
   b. Increase by 2.5% 
   c. Decrease by 5% 
   d. Decrease by 17.5%.

3. a. \( V = 6250(1.085)^n \)
   b. \( V = 14.92(1.0225)^n \)
   c. \( V = 115000(1.11)^n \)

10.1

1. \( M = 4.2 \times 31.62^K \)
2. \( 7.46 \times 10^{11} \) MJ
3. \( 7.46 \times 10^{14} \) MJ

4. A ‘quake measuring two more on the Richter scale releases 1000 times as much energy!

10.2

1. A multiplier of 0.85 means depreciation by 15% per year.
   
2. \( V = 112500 \times 0.85^t \)
3. a. $22148.37
   b. $12540.30

4. The 20th year.

10.3

1. \( b^4 = 0.89 \)
   \[ b = 0.971 \]
   This represents decay of 2.9% per day.

2. \( W = 100 \times 0.971 \)
3. a. 81.38 grams
   b. 0.0022 grams

4. 
   \( 0.971 = 0.5 \)
   \[ \Rightarrow t = 23.6 \]

10.4

1. 1.259
2. \( I = 10^{-12} \times 1.259^d \)
3. a. \( 4 \times 10^{-7} \)
   b. 0.0318

10.5

1.
   \( 1000 \times 1.015^{200} = 19643.03 \)

2. \( V = 1000 \times 1.005^f \)
3. \$19935.90
4. \( V = 1000 \times 1.00164^f \)
   \$20080.59

5. Value compounded quarterly = $2.44
   monthly = $2.61
   daily = $2.714567
   second = $2.718281
   Compounded continuously the value is \( e = 2.718281828459... \)
11.1

1. The flow decreases rapidly at first, then more slowly. Given the context the flow should decay to zero.

2. \( F = 51.7 \times 0.883^t \)

3. \( F = 51.7 \times 0.87^t \)

4. \( t = 16.77 \) i.e. about three quarters the way through October 1999.

5. The increase in flow suggests the well was ready in November.

6. \( F = 31.5 \times 0.905^{t-10} \)

When \( t = 29 \), \( f < 5 \) i.e. November 2001.

11.2

1. Adders in \( n = 1 \)
Multipliers in \( H = \ldots \)

<table>
<thead>
<tr>
<th>( n )</th>
<th>( H )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.50</td>
<td>0.9369</td>
</tr>
<tr>
<td>0.51</td>
<td>0.9796</td>
</tr>
<tr>
<td>0.52</td>
<td>0.8353</td>
</tr>
<tr>
<td>0.53</td>
<td>0.5221</td>
</tr>
</tbody>
</table>

2. \( F = 10 \times 0.849^n \)

3. As \( n \), the bounce number, must be a positive integer, no additional heights can be interpolated between bounce 0 and bounce 14.

4. \( y = 78 \times 1.027^x \)

5. 151.8

11.3

1. \( W = 0.93 \times 1.34^t \)

When \( t = 29 \), \( f < 5 \) i.e. November 2001.

But this model fits the data very poorly.

11.4

1. \( 103.2 \div 1.026^{10} = 79.8 \)

Hence \( y = 79.8 \times 1.026^x \)

2. Because there is no constant adder in internet use.

Because the scatter plot is curved!

3. The "best" simple exponential model that can be obtained is something like \( W = 0.93 \times 1.34^t \)
The reason that there is no better model of this type is that the data does not have an approximate constant multiplier.

4.

5. In 2012.

6. There is a good chance that the prediction will be inaccurate as there may not be another 13% of the world’s population ready to take up internet use (i.e. the trend of increasing growth may be unable to continue).