NSW GENERAL MATHEMATICS Preliminary Course

Mathematical Interactions

Algebraic Modelling

Barry Kissane
Anthony Harradine

Mathematical Interactions:
Algebraic Modelling

Published by Shriro Australia Pty Limited
72-74 Gibbes Street,
Chatswood, NSW 2067, Australia

A.C.N. 002 386 129

Telephone: 02 9370 9100
Facsimile: 02 9417 6455
Email: casio.edusupport@shriro.com.au
Internet: http://www.school.casio.com.au

Typeset by Peter Poturaj of
Haese & Harris Publications

ISBN 1 876543 54 X

About this book

Calculators are too often regarded as devices to produce answers to numerical questions. However, a graphics calculator like the Casio CFX-9850GB PLUS is much more than a tool for producing answers. It is a tool for exploring mathematical ideas, and we have written this book to offer some suggestions of how to make good use it when exploring ideas related to algebra.

We assume that you will read this book with the calculator by your side, and use it as you read. Unlike some mathematics books, in which there are many exercises of various kinds to complete, this one contains only a few 'interactions' and even less 'investigations'. The learning journey that we have in mind for this book assumes that you will complete *all* the interactions, rather than only some. The investigations will give you a chance to do some exploring of your own.

We also assume that you will work through this book with a companion: someone to compare your observations and thoughts with; someone to help you if you get stuck; someone to talk to about your mathematical journey. Learning mathematics is not meant to be a lonely affair; we expect you to interact with mathematics, your calculator and other people on your journey.

Throughout the book, there are some calculator instructions, written in a different font `(like this)`. These will help you to get started, but we do not regard them as a complete manual, and expect that you will already be a little familiar with the calculator and will also use our *Getting Started* book, the *User's Guide* and other sources to develop your calculator skills.

Algebraic Modelling is one of the topics in General Mathematics, mainly because it is a fundamental idea in mathematics and in the applications of mathematics to the real world. Using algebra, we can build models of everyday situations and then use the models to help answer questions we may have. Producing tables of values, graphs and solving equations are all useful to this end. You will experience how the graphics calculator is a useful tool to aid us in these processes.

Although we have sampled some of the possible ways of using a graphics calculator to learn about this topic, we have certainly not dealt with all of them.

We hope that you enjoy your journey!

Barry Kissane

Anthony Harradine

Table of contents

Building and using algebraic models (linear)

Algebra can be used to describe, in an efficient manner, situations that contain quantities that* change, *often called* variables. *The beauty of algebra is that it can describe* all *the possible cases of a situation that arise from the quantities changing. Once an algebraic description (or* algebraic model*) has been formed, we may use it to answer questions of interest about the situation.

Change is one of the fundamental concepts of life. Some forms of change are designed by humans and others by mother nature. Humans design change that has the highest level of predicability. Most situations you will study in this book are examples of human designed change.

Algebraic models may be *formed* in a number of ways. One way is by simply understanding how the change occurs within a situation. The following example is a case in point.

The *cost* of a single mobile phone call is variable. The longer the phone call the larger the cost of the call. The other variable involved here is *time*.

Time changes independently of cost, and hence time is called the *independent variable*. The change in cost depends on the change in time and hence cost is called the *dependent variable*. The *roles*, dependent and independent, can be assigned in many situations where two variables change in a related manner.

The costing involved with mobile phones is very complicated. Many different rates apply for different situations.

We are going to look at how the cost of a single call, as charged by two different companies, changes depending on the length of the call. The type of call is a call made from a mobile during peak time to any phone other than a mobile serviced by the same company as the one making the call. The cost per call, as stated on each company's web site, is as follows:

Company A:	65 cents per 30 seconds
Company B:	$1.20 per minute, plus a flag fall of 15 cents

Interaction A

1 Determine the cost of a six minute long call as charged by
   a) Company A.
   b) Company B.
2 Repeat part 1 for a fifteen minute long call.
3 Repeat part 1 for a thirty and one quarter minute long call.
4 Explain, in words, how you calculated the cost of the calls in part 1, 2 and 3.
5 Make a table of call costs for call lengths of zero to 4 minutes, with increments of 30 seconds.

Your answer to part 4 of **Interaction A** should have been something like this:
For Company A, I multiplied the length of the call (in minutes) by 1.30, *which gave me the cost in dollars.*
For Company B, I multiplied the length of the call (in minutes) by 1.20 *and then added* 0.15, *which gave me the cost in dollars.*

Note that these processes can be used for parts 1, 2 and 3 and are the processes that you would use *no matter the length of the call*. We can *generalise* the calculation of the cost of a call using algebra. We can make an algebraic model (also called a *rule* or *function*) that will determine the cost of a call of *any* length.

Company A

Let C_A be the cost of a call (in dollars) and t be the length of the call (in minutes). The subscript A is a convenient shorthand method of representing Company A.

As the cost of a call equals $1.30 multiplied by the length of the call, then

$$C_A = 1.30t$$

Company B

Let C_B be the cost of a call (in dollars) and t be the length of the call (in minutes). The subscript B is a convenient shorthand method of representing Company B. As the cost of a call equals $1.20 multiplied by the length of the call plus 15 cents, then

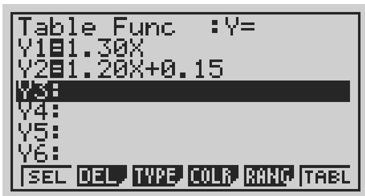
$$C_B = 1.20t + 0.15$$

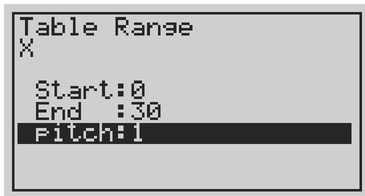
We could produce a table of call costs, for each company, to display some individual cases. For example:

t (mins)	0	0.5	1	1.5	2	2.5
C_A (\$)	0	0.65	1.3	1.95	2.6	3.25
C_B (\$)	0.15	0.75	1.35	1.95	2.55	3.15

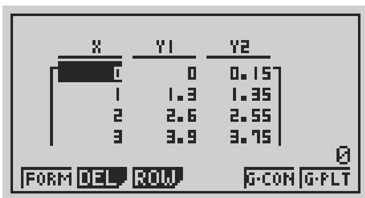
The calculator can be used to display, in an very efficient manner, *many specific cases* for each of the *cost functions.*

The calculator uses X for the independent variable. So, in this case, we need to use X in place of t. In TABLE mode define 1.30X and 1.20X + 0.15 as Y1 and Y2 respectively, where X is entered using the X,θ,T key. Select Y2 and use COLR (F4) and then Orng (F2) to make Y2 orange in colour. Use RANG (F5) to set the calculator to display the cost of calls of whole minute duration from 0 minute to 30 minutes.

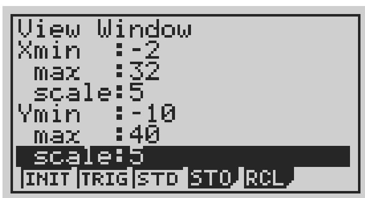




Press EXIT and then use TABL (F6) to produce the table of call costs. You can use the arrow keys to move down and up the table. Notice that the calculator table agrees with the table above.



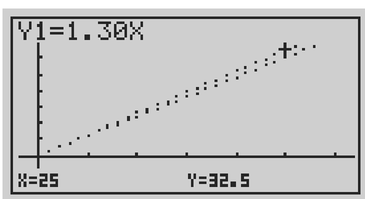
Notice that as the independent variable (length of call) grows by a *constant adder* (which we have chosen to be 1 minute for simplicity), the dependent variable (cost of call) also grows by a ‘constant adder’ (of 1.3 in the case of Company A and 1.2 in the case of Company B). Relationships with this feature are said to be *linear*. Why?

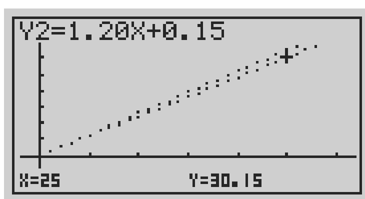


We can also produce a graph of C for each company. First, set the axis end points and scales using V-WIN (SHIFT then F3). Set the values as shown opposite. The choice of Ymin as -10 will ensure the horizontal axis is seen on the screen. Clearly it is not sensible for a phone call to cost a negative amount.

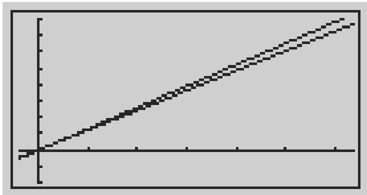
Press EXIT and then produce the table again (F6).

Use G·PLT (F6) to draw the graph to illustrate the cost of whole minute phone calls. You can then use TRCE (SHIFT then F1) and the arrow keys to explore the graphs. Use the up and down arrow to change between the two plots. Note the values for each variable are shown at the bottom of the screen and the function is shown at the top.

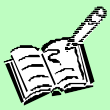




It should now be obvious why such functions are called linear. The points fall in a straight line when plotted. The line has a *constant gradient* or *slope*. The gradient is defined as the change in the dependent variable per *unit* change in the independent variable (or y step on x step). This is also known as the *rate of change* of y with respect to x. In our example the gradients (or rates of change) are \$1.30 per minute and \$1.20 per minute respectively. These values can also be called rates of *charge* in this example.



If we assume that the phone calls are charged in a continuous manner, that is if the call takes only part of a minute then you only pay for part of a minute and not the full minute, then it would make sense to draw a line on the graph rather than just plot points. Use `G·CON (F5)` when you have the table visible to draw such a graph.



Interaction B

1. Use `G·CON` when drawing the plots for the mobile phone calls instead of `G·PLOT`. Experiment with tracing each line. What difficulties did you find?
2. Use the table of call costs or trace the graph to find the cost of a call 15 minutes long as charged by both companies.
3. Change the pitch, in the table range screen, to 0.25. Describe what the table that results will display.
4. Determine how long you could stay on line, with each company, if you wanted to spend \$5.85 on a single call.
5. Determine how long you could stay on line, with each company, if you limited yourself to \$5 for a single call.
6. Which company provides the better deal? Explain your answer.

We could answer question 5 of Interaction B more simply using a little bit of algebra. We could form two linear equations and solve them as shown below:

Company A

$C_A = 1.30t$ and if the cost is to be no more than \$5, let C_A be \$5

$$\Rightarrow \quad 5 = 1.30t$$
$$\Rightarrow \quad 1.30t = 5$$
$$\Rightarrow \quad \frac{1.30}{1.30}t = \frac{5}{1.30}$$
$$\Rightarrow \quad t \approx 3.8$$

So we could only stay on line for 3.8 minutes (correct to 1 decimal place)

Company B

$C_B = 1.20t + 0.15$ and if the cost is to be no more than \$5, let C_B be \$5

$$\Rightarrow \quad 5 = 1.20t + 0.15$$
$$\Rightarrow \quad 1.20t + 0.15 = 5$$
$$\Rightarrow \quad 1.20t + 0.15 - 0.15 = 5 - 0.15$$
$$\Rightarrow \quad 1.20t = 4.85$$
$$\Rightarrow \quad \frac{1.20}{1.20}t = \frac{4.85}{1.20}$$
$$\Rightarrow \quad t \approx 4.0$$

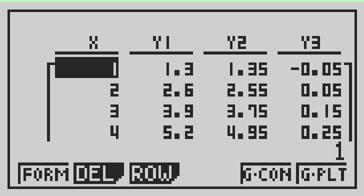
So we could only stay on line for 4 minutes minutes (correct to 1 decimal place)

Interaction C

1. Form two linear equations and solve them to determine the length of time, with each company, for which one could stay on line and spend \$20. Verify your solutions using your calculator.
2. Use the calculator to determine the difference between the cost of an 8 minute call as charged by Company A and an 8 minute call as charged by Company B.
3. Use the calculator to determine the difference between the cost of a 15 minute call as charged by Company A and a 15 minute call as charged by Company B.
4. Use the calculator to determine the difference between the cost of a 25 minute call as charged by Company A and a 25 minute call as charged by Company B.
5. Show that the difference (d) in the cost of calls of the same duration (cost by Company A – cost by Company B) can be calculated by the function

$$d = 0.1t - 0.15$$

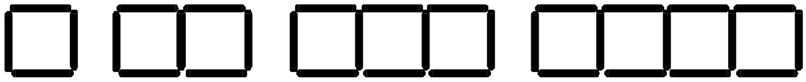
6. Define $0.1t - 0.15$ as Y3 in TABLE mode of the calculator (do not delete the other functions already defined) and produce a table of call costs for each company *and* the difference of call costs as seen below. Verify that your answers to parts 2, 3 and 4 are correct.



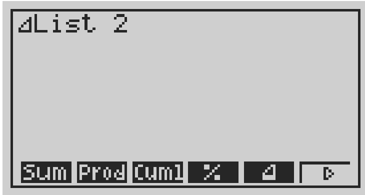
7. Which of the two companies seems to be the better option based on this simple analysis of mobile phone costs?

Another way to form an algebraic model is by starting with some numbers and finding a rule that generates those numbers. A simple example like a stick pattern illustrates this technique well. For example, suppose you were making a line of squares using toothpicks.

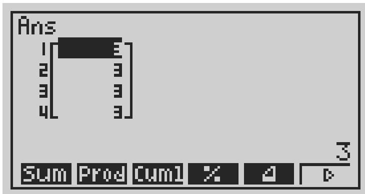
It takes four toothpicks to make one square, two squares need seven toothpicks, three squares require ten toothpicks… and so on.



Information like this can be stored into lists to look for patterns in how the numbers change. Enter the LIST mode of the calculator. The screen here shows the number of squares (1 up to 5) in List 1 and the number of toothpicks needed in List 2. Enter this data.



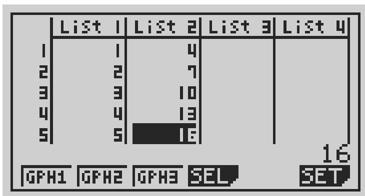
One kind of pattern here concerns the change in number of toothpicks needed for each extra square. You can get difference patterns like this on the calculator using a special difference (Δ) command in the List menu. In RUN mode, press OPTN and activate the LIST (F1) menu and then the difference command (F6 then F6 then F5) followed by 2, to get the differences between each term in List 2.



The screen shows that each difference is 3, confirming that each extra square needs three extra toothpicks.

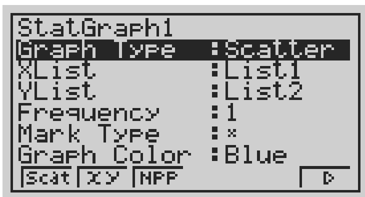
This process would not be sensible, of course, if the data were not in order or the sequence of numbers had some missing values.

One use of patterns is to make, and understand, predictions. For example, suppose we wanted to know how many toothpicks would be needed to make six squares. Since each extra square needs three extra toothpicks, and five squares needed 16 toothpicks, we can see that 19 toothpicks would be needed.



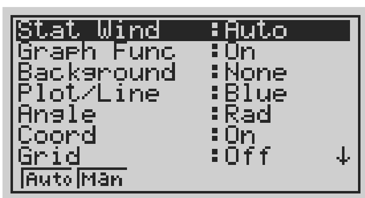
What if we wanted to make 100 squares, however? A different way of looking at the pattern may help.

Enter the STAT mode and press GRPH and then SET to set up a scatter graph in StatGraph1 for the data in List 1 and List 2.

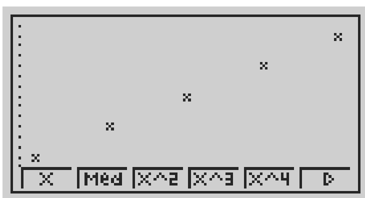


These are the settings needed to make the scatter graph.

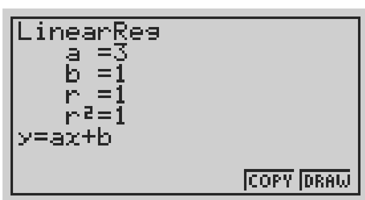
Press EXIT when you have entered these settings



Use SETUP (SHIFT then MENU) to check that Stat Wind is set to Auto. This will ensure that the calculator automatically chooses a sensibly scaled set of axes for this data.

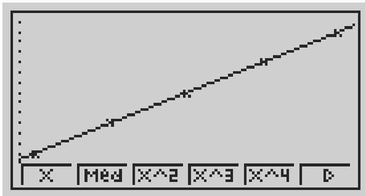


Press EXIT and then use GPH1 (F1) to draw a scatter graph. The points appear to form a line.



To check that the points do in fact form a line exactly, press X (F1) which will give us the rule for the line of best fit and tell us if the points form a perfectly straight line.

The screen shows that the line $y = 3x + 1$ fits the data exactly, since $r = 1$. This (r) is the correlation coefficient. You will learn more about the correlation coefficient next year. If r is reported as 1, we have a pattern that is exactly linear.



You can also DRAW (F6) a line through the scatter plot to illustrate the linear nature of the points. The screen shows that the graph goes through all of the data points.

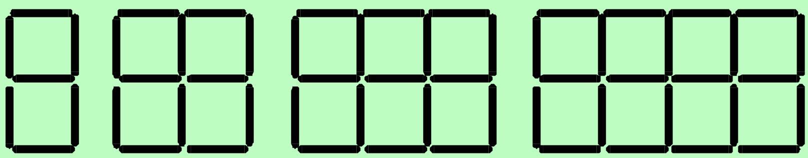
Note, however, that the points on the line in between our data points have no meaning in this context.

The rule, $y = 3x + 1$, allows you to predict how many toothpicks (y) are needed to make various numbers of squares (x).

So, to make 100 squares, you will need $y = 3(100) + 1 = 301$ toothpicks.

Interaction D

1. Explain, with a diagram, why each extra square in the pattern needs three extra toothpicks.
2. According to the pattern developed above, how many toothpicks would be needed to make 8 squares? Check your answer by using toothpicks (or drawings of them) to make the squares.
3. Instead of a single story line of squares let us consider a two story line of squares. Each diagram is considered as a level, the next level being produced by adding two more squares.



   Determine the rule that links y and x, where y is the number of toothpicks required and x is the level number. Use your rule to determine how many toothpicks would be needed to build level 200.
4. Explain why the rule $y = 2(3x + 1)$ would seem sensible for the pattern found in question 3 at first glance, but is in fact wrong.

Investigation:

If Company A, from our mobile phone example, was to reduce its rate to 50 cents per minute, how would the graph of cost of call by time change? If Company B increased its flagfall to 25 cents, how would the graph of cost of call by time change?

Using a step function

The algebraic models in the previous section are useful for understanding some aspects of the costs of mobile phone calls. They are, however, a little less complicated than what happens in practice – which is often the case for models. We will now look at a more realistic pair of models.

Consider what would happen if the mobile phone companies changed the way they charged ever so slightly to:

Company A: 65 cents per 30 seconds *or part thereof*

Company B: $1.20 per minute, *or part thereof*, plus a flag fall of 15 cents

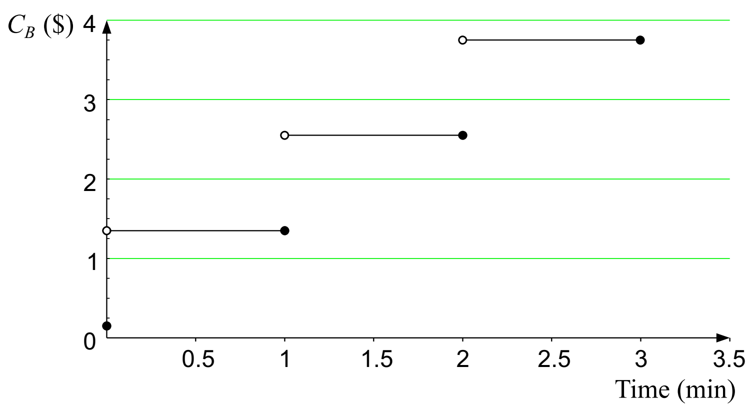
The phrase *or part thereof* changes things a little. No longer can we use a continuous linear function to model the cost of a call. In this situation, unlike before, the cost of a 4-minute call with Company B is the same as a three and one half minute call or a 3.2-minute call or any length of call greater than 3 minutes but less than or equal to 4 minutes. Such situations can be modelled using a *step* function.

A table of call costs can be formed for Company B.

t (mins)	0	0.5	1	1.5	2	2.5
C_B ($)	0.15	1.35	1.35	2.55	2.55	3.75

A graph that displays the table of values would look as follows:

The step like image gives this type of function its name.



Interaction E

1. Make a table of values and draw a graph, on graph paper, that represents how the cost of a call will vary, for a length between zero and three minutes (show 15 second intervals on your table and graph), as charged by Company A.
2. Draw the graph of the step function for Company B on the same set of axes that you used for Question 1.
3. Determine the charge for a twelve and one quarter minute phone call as charged by both companies.

Working out the rule for a particular step function is sometimes a tricky business. For the two company charges, the rules are:

Company A: $C_A = -0.65\,[-2t]$

Company B: $C_B = -1.20\,[-t] + 0.15$

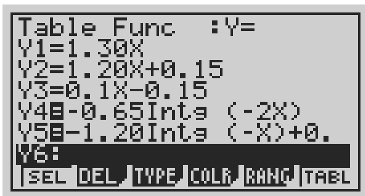
The [ ] brackets refer to a function called the *greatest integer function*.

This means: find the greatest integer that is less or equal to the number in the brackets. So, [3.18] = 3 and [4.7] = 4

When entering the greatest integer function into the calculator, the notation used is not the [t] brackets but the `Intg (t)`. Therefore the rules become:

Company A: $C_A = -0.65$ `Intg` $(-2t)$

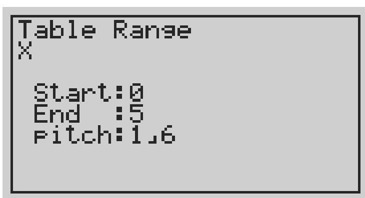
Company B: $C_B = -1.20$ `Intg` $(-t) + 0.15$

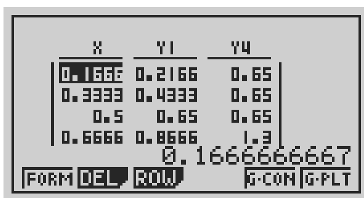


In `TABLE` mode define `-0.65 Intg (-2X)` and `-1.20 Intg (-X) + 0.15` as `Y4` and `Y5` respectively. You can access the Intg command by pressing `OPTN` then `NUM (F5)` and finally Intg `(F5)`. You will need to have defined `Y1` and `Y2` as shown opposite.

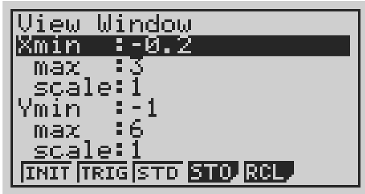
Note that three of the equal signs appear different. The dark square around the equal sign of `Y4` and `Y5` means they are the only ones for which a table will be produced. To achieve this, highlight each function in turn, and use `SEL (F1)` to de-select each of them. They will be de-selected when their equal sign does not have a dark square around it. It is sometimes best to de-select rather than delete functions as you may need to return to them later. To delete unwanted functions, highlight them and use `DEL (F2)`.

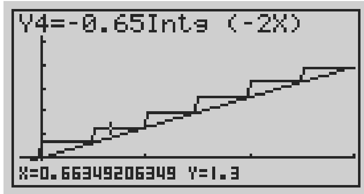
Use `SEL (F1)` to select `Y1` and `Y4`. Use `RANG (F5)` to set the values as shown below. Press `EXIT` and then use `TABL (F6)` to produce a table of `Y1` and `Y4` values. Check that it agrees with the tables you completed earlier.





Set the viewing window parameters as shown below and then use `TABL (F6)` and then `G·CON (F5)` to produce a graph of the call cost by time. You can trace these functions, as shown in the on the next page.





Notice that both the table and graph illustrate how the two different charge schemes that we have studied, for Company A, differ. Note also that, when G·CON is used, the calculator does not draw a step function quite correctly.

Interaction F

1. In GRAPH mode use SETUP (SHIFT then MENU) to set the Draw Type to Plot. Press EXIT and use SEL (F1) to select Y4. Use Draw (F6) to draw the graph of Y4. Explain how this graph of Y4 differs from the ones produced by G·CON and G·PLT in TABLE mode.
2. Which of the three graphs looks most like the conventionally correct graph seen on page 12?
3. What is misleading, with respect to call costs about the graphs of Y4 produced from G·CON and G·PLT?
4. Trace the graph of Y4, produced in GRAPH mode, to see that phone calls greater than 0 but less than or equal to 0.5 of a minute cost $0.65, while those greater than 0.5 of a minute but less than or equal to 1 minute cost $1.30. How is this different from the case for Y1?
5. In GRAPH mode graph Y2 and Y5 together, making them different colours. Explain how the graph displays the difference in the two methods of charge.

Now re-focus specifically on the new charging method for calls made with company A and company B.

Interaction G

1. In GRAPH mode select both of the Intg functions (Y4 and Y5) and make them different colours. Use DRAW to produce two graphs on the same axes. Explain how the graphs illustrate the difference in the two charging methods.
2. Use a table of call costs or trace a graph to find the cost of a call 15 minutes long as charged by both companies.
3. Determine how long you could stay on line, with each company, if you wanted to spend no more than $5.85 on a single call.
4. Determine how long you could stay on line, with each company, if you wanted to spend no more than $5 on a single call.
5. Which company provides the better deal under these conditions? Explain your answer.

Investigation:

Visit the web sites of at least two mobile phone call providers and explore the complexities that determine the amount of the bill a mobile phone owner receives.

Exploring variables

***Variables** (quantities that can vary) are usually represented by a single letter. It is important that you understand that a letter, when used in this way, is taking the place of* **any** *number.*

For example, the time it takes you to eat your lunch is a variable, since it will be different on different occasions.

Your calculator represents variables with capital letters $A, B, C, \ldots$. In RUN mode, you can store a specific value for a variable by using the arrow key, just above the ON key. To get the letters on the screen, you need to press the pink ALPHA key first. The calculator can store only one value for a given variable at a time.

```
13→A
                 13
A+2
                 15
```

In this screen, a value of 13 has been assigned to the variable A.

Then notice that $A + 2$ has the value of 15.

The calculator keeps the same values for variables after you clear the screen or even turn the calculator off.

```
2A-1→B
                 25
B
                 25
B²-1
                624
```

When $2A - 1$ is assigned as the value for the variable B, it then has the value 25.

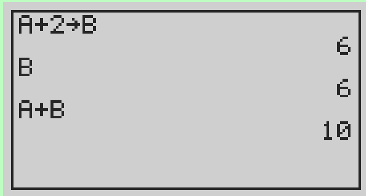
Notice that, as in algebra, the calculator interprets $2A$ to mean $2 \times A$ and B^2 to represent $B \times B$.

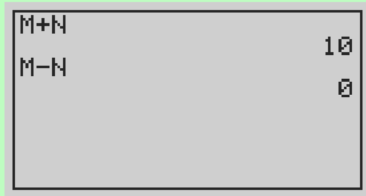
Interaction H

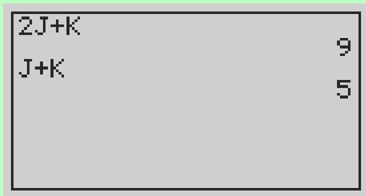
1. Work out what the values of P and Q must have been for this screen to be produced in RUN mode. Explain your thinking to your partner. Then use your answer to produce a screen exactly like this one.

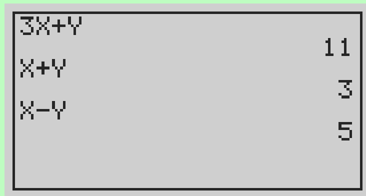
```
P+2
                  5
3P
                  9
P+Q
                 17
```

2. What values of the variables will give the following screens? Check your thinking by reproducing the screens exactly. Explain your thinking in a few sentences.







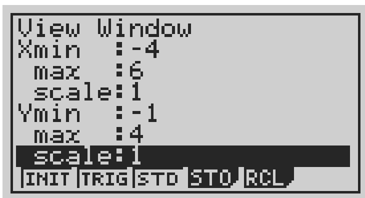


3. Make up some more screens like those in the previous two questions. Show them to your partner to see if they can work out what you set the values of the variables to be.

Viewing calculator graphs

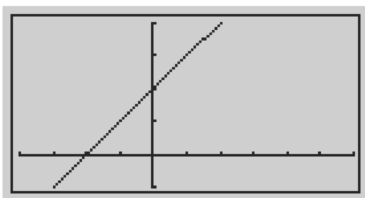
It is important for you to realise that a graph on a calculator depends on the rule used and also on the scales selected on each axis. The same is true for graphs drawn on computers or drawn by hand. So you need to choose scales carefully, and understand their effects.

The screen on your calculator is almost (but not quite) a rectangle that is twice as wide as it is high.



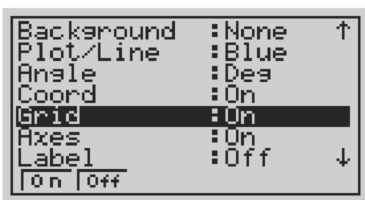
Define Y1 as x + 2 and set the view window as shown opposite.

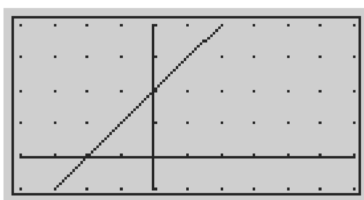
Draw the graph of Y1. Your screen will look like the one following. The graph is a line, sloping up from left to right. The graph intersects the y-axis at (0,2) and the x-axis at (-2,0).



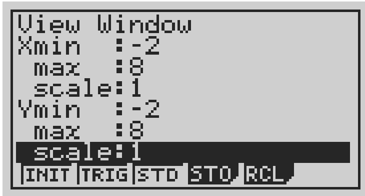
For this graph, it looks as if 1 unit takes up the same distance along each axis. Hence a *square* graph is produced. You may not have expected this to occur given that the distance between the min and max for the X axis is twice that for the Y axis. This is almost what the graph would look like if you drew it on graph paper with a scale of 1 square = 1 unit on each axis. For this particular graph, in fact, the line seems to be at 45 degrees to each axis.

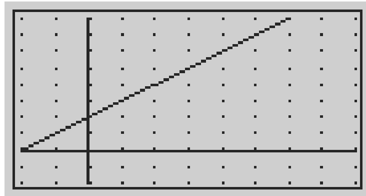
One way of seeing the 'squares' is to turn on the Grid in the Set Up menu, as shown below. Then when you EXIT and draw the graph again, a grid of points is shown corresponding with your scales. In this case, the grid shows small squares.





Now change the view window settings to those seen below.





If we draw the graph of `Y1` again we get the graph shown here.

Note that the 'squareness' of the graph has been lost, as the grid shows. Now you can see small rectangles instead of small squares. The graph is still a line, sloping up from left to right, it still intersects the y-axis at (0,2) and the x-axis at (-2,0).but now the line *appears* to slope differently. So, to interpret the graph on the screen, you need to make sure that you know what scales have been used.

In fact, the screen of the CFX-9850GB PLUS is not *exactly* a 2 by 1 rectangle. It has 127 square pixels left to right and 63 square pixels top to bottom. A ratio of 127 : 63 is very close to, but not the same as, 2 : 1.

The easiest way to choose a scal e for which the units on each axis are *exactly* the same, which is sometimes convenient, is to start with the `INIT` view window. This choice has the extra advantage that convenient steps for tracing a graph are obtained automatically.

Interaction 1

1. Define `Y1` as $x + 1$ to draw a graph of $y = x + 1$. Set the view window so that a square graph is produced. Describe the resulting graph.
2 Change the view window values so that a graph is produced that appears to slope *less* steeply than that produced in question 1. Change the values again to produce a graph that appears to slope *more* steeply than that produced in question 1.
3. Draw graphs of $y = x + 2$ and $y = 1 – x$ together using the same `INIT` view window. Then draw the graphs again, for $–3 < x < 3$ and $–3 < y < 3$. Describe how the relationship between the graphs changes when the scales are changed.
4. Draw a graph of $y = x – 1$ in the `INIT` window. Then choose the `STD` view window and redraw the graph. What aspects of the graph are changed? What aspects remain the same?
5. Draw a graph of $y = x + 1/2$ in the `INIT` view window. (You can use the fraction key to write 1/2.) Trace the graph to find the y-value when $x = 1.5$. Do the same using the view window that we started with in this section and that is exactly a 2 by 1 rectangle: $–4 < x < 6$ and $–1 < y < 4$. Describe any differences you notice.
6. Tayla wants to draw a graph of $y = x$, and expects to get a line at 45 degrees to each axis. Which (if any) of the following choices of scales will achieve this?

(a)	$-4 < x < 4$	and	$-4 < y < 4$
(b)	$-12.6 < x < 12.6$	and	$-6.2 < y < 6.2$
(c)	$-9.4 < x < 9.4$	and	$-3.1 < y < 3.1$
(d)	$-31.5 < x < 31.5$	and	$-15.5 < y < 15.5$

Test your predictions by trying them out. Give another possibility that will achieve the desired effect.

Investigating constant rates of change

The idea of rate of change of one variable with respect to another is an important one in many areas of life. Rates of change can be positive, negative, zero, big or small. When the rate of change is* constant *we have a situation that can be modelled with a linear model (or function) – like the mobile phone calls or the stick pattern.

We will now investigate the effect of the value of a constant rate of change on the graph of the corresponding linear function.

All linear functions have the form

$$y = mx + b$$

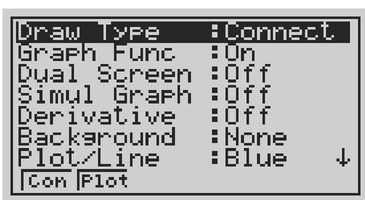
where m is the value of the constant rate of change (or slope/gradient/constant adder) and b is the value of y for $x = 0$ (eg. the flag fall).

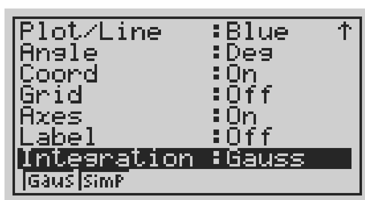
The GRAPH mode allows us to produce graphs of functions without first having to produce a table of values as in the TABLE mode. (The calculator actually does produce a table but does not show it to us.)

We will investigate the function $y = mx + 1$ for a variety of values of m.

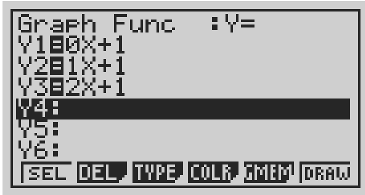
From the MAIN MENU select the GRAPH (5) mode. You may find that some functions exist in this mode already. They will be the ones that you defined in the TABLE mode earlier. Functions defined in TABLE mode will appear in the GRAPH or DYNA modes – or vice versa.

Use SET UP (SHIFT then MENU) to ensure the settings for this mode are as follows:

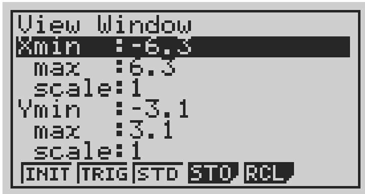




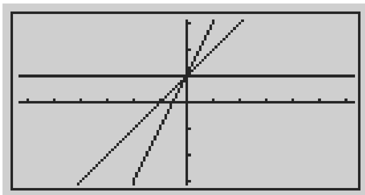
With each function highlighted in turn, either use SEL (F1) to de-select each of them or use DEL (F2) to delete them. We have deleted them.



Define Y1 as $0x + 1$, Y2 as $1x + 1$ and Y3 as $2x + 1$. Before drawing the graph of these functions we must tell the calculator what scales to use for the axes of the graph.



Use V-Window (SHIFT then F3) and then INIT (F1) to set the view window parameters as shown opposite. INIT stands for initial and produces a set of axes with the same scale on both the horizontal and vertical axes and gives a nice result when tracing.



Press EXIT and then use DRAW (F6) to draw the graphs of the three functions.

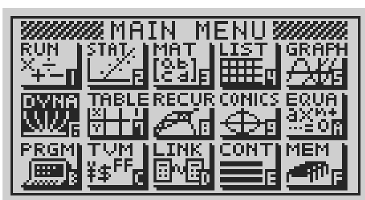
It should be easy for you to distinguish between which line corresponds to which function here. You could, however, make each line a different colour so you could be sure. Tracing the lines will also allow you to check which is which.

Interaction J

1. Define Y4 as -1*x* + 1, Y5 as -2*x* + 1, Y6 as *x* + 1, Y7 as -*x* + 1, Y8 as 4*x* + 1 and Y9 as -4*x* + 1 in the calculator and draw the graphs of them and the ones above all at once. Give the values of *m* and *b* for each function.
2. Describe the effect that changing the value of *m* from a small positive value to a large positive value has on the graph of the function.
3. Describe the effect that changing the value of *m* from a positive value to a negative value has on the graph of the function.
4. If the rate of change is zero, explain why the graph that results is horizontal.

Dynamic Graphing

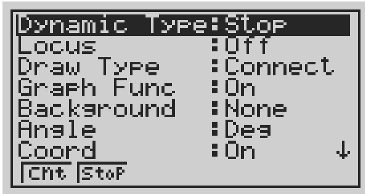
Instead of defining many different functions to test the effects of changing m values, we can use the *Dynamic Graphing* ability of the calculator. We will investigate the function: $y = mx + 1$ for a variety of values of m again.

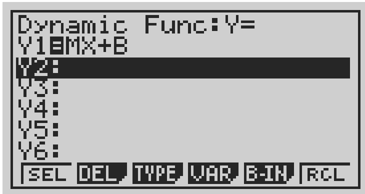


Select the DYNA (6) mode from the MAIN MENU. This is called the *Dynamic Graphing* mode.

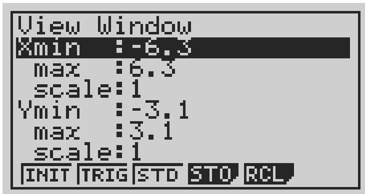
Again, you may see some functions in the function list from previous use. Use DEL (F2) to delete each function.

Use SET UP (SHIFT then MENU) to locate the set-up screen for this mode. Ensure all the settings are as shown below. Press EXIT.



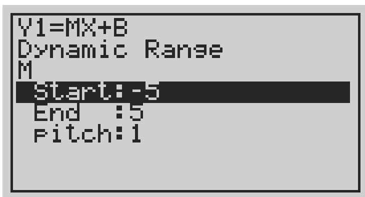


Define Y1 as MX+B. Use the ALPHA key to enter the M and the B.



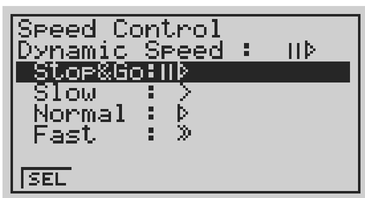
Use V-WIN (SHIFT then F3) to set the axes limits and scale as shown opposite (INIT will do it). Press EXIT when you are done.

Use VAR (F4) to select which value, M or B, you want to vary. Highlight M and press SEL (F1) to select it. Set the value of B to 1 and press EXE to confirm your choice.



Now use RANG (F2) to set the range of the values that you would like to have for M. Set them as shown.

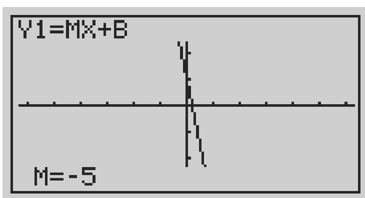
Press EXIT.



Use SPEED (F3) to set the speed at which the different

graphs of Y = MX + 1 will appear. Highlight Stop & Go and press SEL (F1).

Press EXIT.



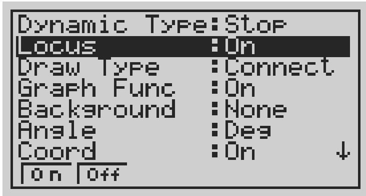
Now press DYNA (F6), and after 'One Moment Please!' a graph will appear – that for M = -5. Press EXE several times to cycle through the values chosen for M.

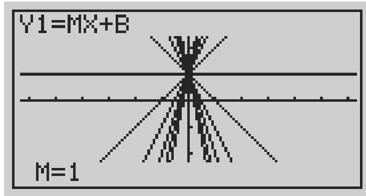
Rather than manually cycle through the M values you can automate this.

Press the AC/ON key and change the speed to slow (F2). Sit back and think! Press AC/ON to stop.

You can see the graphs another way. Press EXIT twice and use SETUP (SHIFT then MENU) to arrive at the set-up screen. Turn the LOCUS option ON. Press

EXIT and then produce a DYNA graph again. The graph below should result. The moving blue line shows the graph for the M value shown.





Interaction K

1. Draw a sketch (by hand) to illustrate the difference between y = 4*x* + 1 and y = 10*x* + 1.
2. Draw a sketch to illustrate the difference between y = -1*x* + 1 and y = -10*x* + 1.
3. Draw a sketch to illustrate the difference between y = 2*x* + 1 and y = -5*x* + 1.
4. Produce the dynamic graph again for Y=MX+B. What do you notice about the angular displacement of the lines as the M value changes by 1 each time.

Investigation:

Once a graph is drawn, SQR (F2) can be accessed by pressing ZOOM and then the continuation key (F6). It will give you a so called square graph (no matter what the view window is set at), one that resembles a hand drawn graph with the same scale on each axis. Hence a slope of 1 will look like a slope of 1 (ie. 1 unit right then 1 unit up) with respect to the axes. It does, however, make the values obtained when tracing not particularly nice. Experiment with the SQR command when you draw graphs and do not start with INIT settings for the view window.

Investigating the flag fall (vertical intercept)

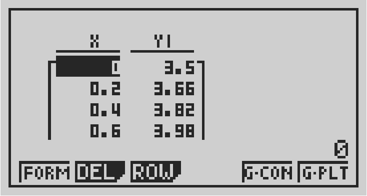
The flag fall is an interesting concept. In our mobile telephone study, Company A chose not to charge a flag fall while Company B charged 15 cents. Essentially a flag fall, in this case, is a charge you pay for no service (the amount a call would cost that had a duration of zero minutes). It may seem hard to justify such a charge in some instances.

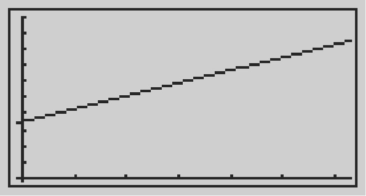
When you use a taxi you are charged a flag fall. When you get into the taxi the meter starts at a figure greater than $0. It is a fixed charge each customer pays to contribute to the costs of maintaining the taxi, regardless of the distance travelled.

Imagine that the flag fall for a taxi is $3.50 and the charge for travelling is 80 cents per kilometre. We could formulate the following relationship:

$$C = 0.80d + 3.50$$

where C is the cost of a trip d kilometres long. A table and graph for this situation produced on the calculator looks as follows.





The graph cuts the vertical axis at the point (0,3.5). Hence the vertical intercept (or the y-intercept) is said to be 3.5, which of course corresponds to the value of the flag fall.

Interaction L

1. Use the DYNA mode to produce a dynamic graph of $y = mx + b$ for $m = 3$ and b values ranging from –5 to 5. You will need to set b to be the dynamic variable.
2. Write down the main difference between a graph of a linear function if b is positive compared to one for which b is negative.
3. How will the graph of $y = mx + b$ change if m is constant and b is changed from 4 to 10?
4. How will the graph of $y = mx + b$ change if m is constant and b is changed from -4 to -10?

Investigation:

Carry out some research to determine how the charge for a taxi trip is actually calculated. Is a continuous linear function an appropriate model? Is distance travelled the correct independent variable? Use the information you find to build an appropriate algebraic model and produce a graph for this model.

Breaking even

***When a business is first set up the owner must outlay an amount of money for which there is no immediate return. This is the money that pays for the initial purchase of machinery, materials and the like. This initial outlay is like a flag fall. Once the business is producing and selling goods, it has other costs as well as the initial costs, but also has an income. Hence the initial outlay can begin to be recouped. Most people are interested to know how much product they need to sell before they* break even *– that is when* income is equal to the initial outlay.**

Consider the following simple business situation.

> *Last holidays Tyron was bored. He decided to start a little business in which he made and sold skateboards.*
>
> *He had to set up his Dad's shed with tools and a bench which cost him \$500. Each board he produced cost him \$30 to make. Once made, he sold them for \$50.*
>
> *He wondered how many boards he would have to sell before he broke even, that is until the income from selling boards equalled the initial outlay* **and** *the cost of making that number of boards.*

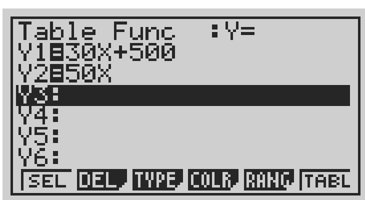
The cost (C) associated with the number of boards (b) that Tyron makes can be modelled by

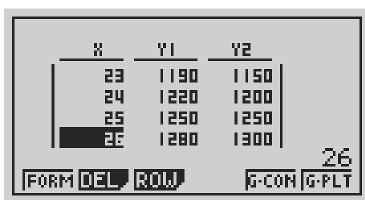
$$C = 30b + 500.$$

The income (I) associated with the venture can be modelled by

$$I = 50b.$$

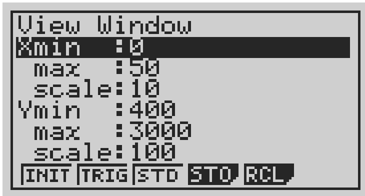
Define both of these functions in the TABLE mode of the calculator. Then produce a table that will allow us to see how many boards Tyron will need to sell to break even. Our table looks like the following.

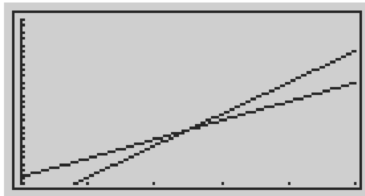




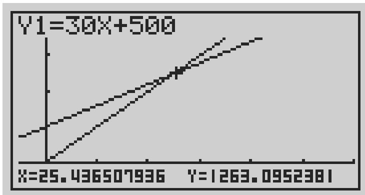
It looks like he must sell 25 skateboards to break even (ie. Y1 = Y2). We could discover this in another way.

Enter the GRAPH mode of your calculator. The two functions just defined will be present. Set the view window of the calculator as shown below. Now draw a graph of the two functions.

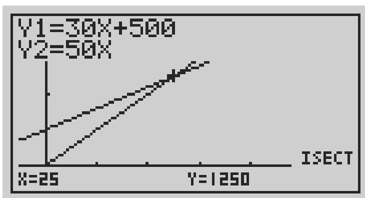




Note that the same values for cost and income that we saw in our table are illustrated on this graph as a cross over point (more formally called the *point of intersection)* of the two lines that illustrate the functions.



We could trace the lines (SHIFT then F1) to find the point of intersection. The accuracy of this method depends on the scale you are using. A more accurate way is to use the calculator function that automatically finds the coordinates of the point of intersection.



With the graph showing on your calculator, use G-Solv (SHIFT then F5) to access ISCT (F5). ISCT stands for intersection. Wait, and a moving cursor travels along one of the lines until it reaches the point of intersection.

Again we see that Tyron would need to make and sell 25 skateboards to break even. Both cost and income are \$1250.

We could also achieve this result by solving an equation. Clearly to break even, income must equal cost, so

$$\begin{aligned} \text{income} &= \text{cost} \\ \Rightarrow \quad 50b &= 30b + 500 \\ \Rightarrow \quad 50b - 30b &= 30b + 500 - 30b \\ \Rightarrow \quad 20b &= 500 \\ \Rightarrow \quad \frac{20b}{20} &= \frac{500}{20} \\ \Rightarrow \quad b &= 25 \end{aligned}$$

You now have three ways to solve a break even problem.

Interaction M

1. Determine how many boards Tyron will have to sell to break even if he sells the boards for:
    - i) $35
    - ii) $40
    - iii) $60
2. What would the selling price need to be to break even after selling just one board?
3. What selling prices will result in Tyron never breaking even? Illustrate these situations graphically.
4. Tyron's dad decides his son is onto a good thing and decides to get into the business of making and selling skateboards. He sets up a factory at the cost of $40 000. He is able to make the boards at the cost of $20 per board. The selling price is set at $60 per board – they are a bit flasher than the Tyron originals. Determine how many boards that Tyron's dad must sell before he breaks even.

Checking equivalence

From the solving of equations you have seen in this book you should have realised that algebraic skills are important. Sometimes it is helpful to replace algebraic expressions with* equivalent *expressions, which have the same value for* all *values of a variable. You will have learned many rules for doing this in your earlier studies of algebra.

For example, consider the (complicated) expression:

$$1 + 5x^2 + 3x + 9 - x^2 - 2x + 11x - 14$$

To find the value of this expression when $x = 4$ requires many separate calculations:

$$1 + 5 \times 4^2 + 3 \times 4 + 9 - 4^2 - 2 \times 4 + 11 \times 4 - 14 = 108$$

It is easier to first change the expression into an equivalent expression. In this case, you can collect together the *like terms* to produce simpler versions. For this expression, there are three sorts of terms, those involving numbers, those involving the pronumeral x and those involving x^2:

$$\begin{aligned} 1 + 5x^2 + 3x + 9 - x^2 - 2x + 11x - 14 &= (1 + 9 - 14) + (3x - 2x + 11x) + (5x^2 - x^2) \\ &= (10 - 14) + (x + 11x) + (4x^2) \\ &= -4 + 12x + 4x^2 \end{aligned}$$

So an equivalent expression to the original one is

$$4x^2 + 12x - 4.$$

To evaluate this expression for $x = 4$ is much easier than before:

$$4 \times 4^2 + 12 \times 4 - 4 = 108.$$

Since there are less calculations to do, it takes less time to do them and it is less likely that you will make a small error.

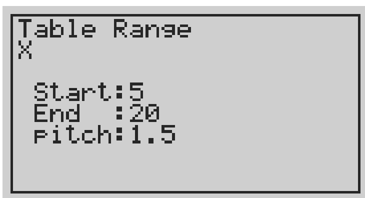
Interaction N

1. See how long it takes you to find the value of the expressions: $1 + 5x^2 + 3x + 9 - x^2 - 2x + 11x - 14$ and $4x^2 + 12x - 4$ for some different values of x. Try $x = 3$, $x = 5$ and $x = -2$. How much time is saved by using the shorter equivalent expression?
2. Find an equivalent expression to $a - 3a^2 + 13a + 6 - a^2 - 8a + 5a - 9$. Test your result with three different values of the variable a.
3. Check to see if $5 - t + 11t^2 - 4t + 12 - 9t^2 = 2t^2 + 5t + 17$. (If you conclude that the two expressions are *not* equivalent, change the simpler one so that they *are* equivalent.)

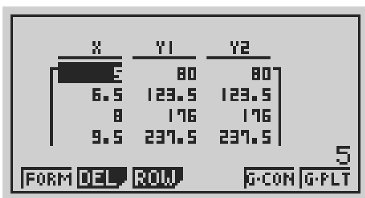
Your calculator will not produce equivalent expressions for you – you will have to do this yourself, using the rules you may have learned before (such as collecting the like terms, using the distributive property, and so on). However, you *can* use your calculator to test whether two expressions seem to be equivalent or not.

For example, consider the two expressions $2x(x + 3)$ and $2x^2 + 6x$. If these are equivalent, they should have the same value for *any* value of the variable x.

One way to test lots of values of the variable is to use a table of values. Enter the TABLE mode. Use DEL (F2) or SEL (F1) to delete or to de-select any functions already showing. Then enter the two expressions into the function list, using the X,θ,T key for the pronumeral X. The screen here shows the two expressions defined as Y1 and Y2.



To construct the table, you also need to choose some values for the variable, x. The screen here shows values from 5 to 20, going up in steps of 1.5.

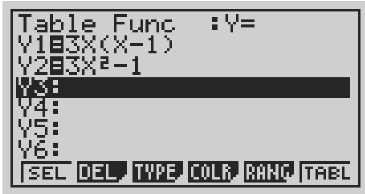


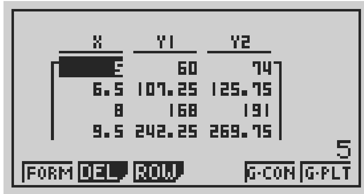
Construct the table, using TABL (F6). The screen at left shows that the two expressions have the same value (seen in columns Y1 and Y2) for each value of x. You can use the down arrow to check the values of each function for other x values not yet shown on the screen. Try to find an x value for which the functions values are different.

So it seems *likely* (since you could not find an x value for which the function's values were different, but have not tried *all* x values) that the expressions are equivalent, that is, $2x(x + 3) = 2x^2 + 6x$.

However, to make *sure* that they are equivalent, you will have to use your algebraic skills (especially the distributive property).

The screens on the next page show another example, to test the equivalence of $3z(z - 1)$ and $3z^2 - 1$. Notice that the calculator can only work with x as a variable, so you need to test the equivalence $3x(x - 1) = 3x^2 - 1$ instead.

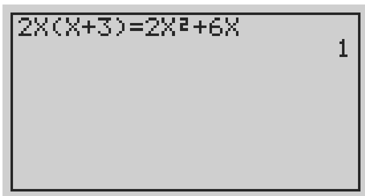


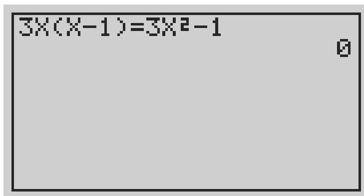


In this case, the screens clearly show that the two expressions in columns Y1 and Y2 have different values for a given value of x. So the expressions are *not* equivalent.

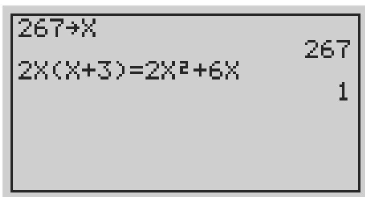
Your calculator cannot be used to prove that two expressions *are* equivalent; but it *can* be used to prove that two expressions are *not* equivalent. If you find *even one* value for which the two expressions are not equivalent, then that is enough to prove that they are *not* equivalent.

You can also use RUN mode of your calculator to make a *quick check* on whether two expressions are equivalent. The two screens below show how to do this, using the equals sign (SHIFT and decimal point). When you enter two expressions with an equals sign in between, the calculator will tell you whether the equality is true (by printing a 1) or false (by printing a 0).



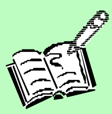


The calculator only checks one value of x (the value previously assigned to X), however, so it cannot always be trusted to tell you that expressions *are* equivalent; as before, it *can* be trusted to tell you when the expressions are *not* equivalent.



One way to make the calculator check more trustworthy is to first assign a 3-digit integer as the value of the variable used, as the screen opposite shows.

This will make it extremely unlikely (but still not impossible) that your calculator will report that two expressions are equivalent when in fact they are not.



Interaction 0

1. Use a table to decide whether or not $(x + 3)^2 = x^2 + 9$. Explain your result fully in a few sentences.
2. Jarryd wasn't sure of the result of expanding $5x(2x + 1)$. He got a result of $10x^2 + 5x$, but his friend Meg got $10x^2 + 1$. Then, to make matters worse, Peter claimed that the answer was $15x^3$. Who was correct? (Or are they all wrong?) Give the correct result and explain any errors the students have made.

3. Notice that $(x - 1)^2 = x^2 - 2x + 1$
$(x - 2)^2 = x^2 - 4x + 4$
$(x - 3)^2 = x^2 - 6x + 9$
$(x - 4)^2 = x^2 - 8x + 16$

Use these observations to predict the expansions of $(x - 5)^2$ and $(x - 7)^2$ and $(x - a)^2$. Check your predictions on the calculator.

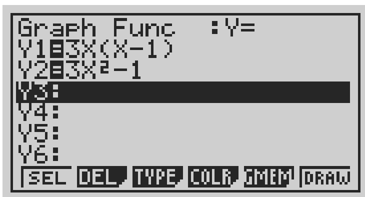
4. Kaye was confident that the result of multiplying $3x^2$ by $5x$ was $8x^3$, but the answer in the back of her algebra book was $15x^3$. Use your calculator to check these two answers and decide which one is correct.

5. Assign zero as the value for X (using the arrow key, which is right above the ON key). Then explain the results of entering the following equalities on the calculator:

$$2x \times 5x = 10x^2$$
$$6x^3 \times 2x = 12x^3$$
$$x(x - 2) = -2x^2$$

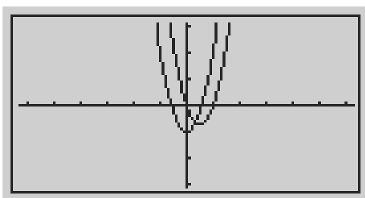
Another way to check to see whether two expressions are equivalent is to use a graph. If two expressions are equivalent, they will have the same values for all values of the variable, and so they will have the same graph.

Enter `GRAPH` mode and then use `SETUP` (`SHIFT` then `MENU`) to check that the `Draw Type` is set to `Connect`. Press `EXIT` and then delete or de-select any functions in the function list, in the same way as for `TABLE` mode.



Consider again the two expressions $3x(x - 1)$ and $3x^2 - 1$.

To see if they are equivalent, first define a pair of functions using the two expressions, as shown in the screen. Make the graph of the second function orange.



Use `V-Window` (`SHIFT` then `F3`) to choose the `INIT` settings for the scaling of the axes.

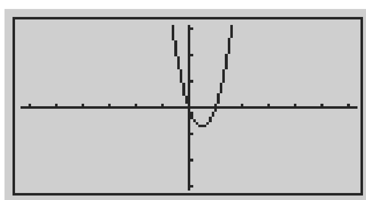
Draw graphs of the two function, using `DRAW (F6)`. The screen here shows that the two graphs are different for most values of x.

The two graphs have the same value for only one value of x (in fact, $x = 1/3$).

So, the expressions cannot be equivalent for all x.

In this case, you should realise that $3x(x - 1) = 3x(x) - 3x(1) = 3x^2 - 3x$. So, the two expressions $3x(x - 1)$ and $3x^2 - 3x$ are equivalent. That is, $3x(x - 1) = 3x^2 - 3x$.

If these two expressions are graphed, as shown below, with the second graph again coloured orange, both graphs will be the same. This illustrates the equivalence of the expressions.



Watch the screen carefully to see that the second (orange) graph falls exactly on the first graph. If you did not know the expressions were the same to start with, drawing graphs is a way to tell for sure that they are different or get an indication that they may be equivalent.

Interaction P

1. Use a graph to decide whether or not $(x + 1)^2 = x^2 + 1$. Explain your result fully in a few sentences.
2. Notice that $(x + 1)^2 = x^2 + 2x + 1$
   $(x + 2)^2 = x^2 + 4x + 4$
   $(x + 3)^2 = x^2 + 6x + 9$
   $(x + 4)^2 = x^2 + 8x + 16$
   Use these observations to predict the expansion of $(x + 5)^2$ and $(x + 7)^2$. Check your predictions by using graphs on the calculator. Start with the INIT screen but change the Y scale to go from 0 to 50.
3. Use a pair of graphs to check that $x^2 + 2x + 4 = (x + 1)^2 + 3$.
   Use this result to complete the equivalence $x^2 + 2x + 7 = (x + 1)^2 + ?$.
   Check your guess by graphing.
4. Draw a pair of graphs to verify that $x(x + 1) - x(x - 1) = 2x$. Then expand the expression on the left of the equals sign to prove that these two expressions *must* be equivalent.
5. To check whether or not $(x+2)(x+2) = x^2 + 4$, Jennifer defined each expression as a function and drew a graph of each using the INIT setting for the scaling of the axes. She was in a hurry and forgot to use different colours for each function. Her conclusion was that the two expressions were equivalent. Do as Jennifer did – was her conclusion correct?

Investigation:

One more than the product of any four consecutive integers can be expressed algebraically as $x(x + 1)(x + 2)(x + 3) + 1$. Use the table to investigate the values of this expression, no matter what the value of x. Find an equivalent expression for $x(x + 1)(x + 2)(x + 3) + 1$ which illustrates the special feature of all its values.

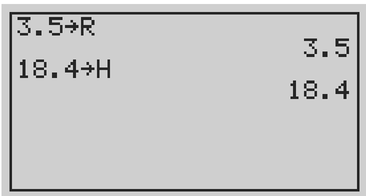
Using formulae

Many other types of algebraic models exist apart from linear models. Some are very complex, are quite hard to make and may involve more than two variables – but actually perform very simple jobs that we can all understand. Sometimes referred to as a **formula,** ***we will look at some and see how the calculator can be used effectively when we work with them.***

The most common kind of formula shows how the value of one variable is related to others. For example, a formula for the volume of a cylinder is

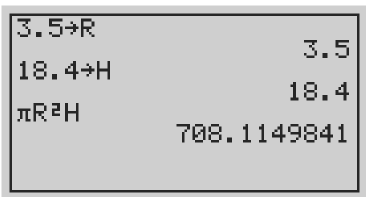
$$V = \pi r^2 h.$$

In this formula, V stands for the volume of the cylinder, r stands for the radius of the base of the cylinder and h stands for the height of the cylinder.



Your calculator can be used to give a value to a variable so that it can be used in a formula. The screen here shows the value of 3.5 being given to r, the radius of the cylinder, and 18.4 being given to h, the height.

Notice that the calculator memories are labelled with capital letters.



The volume can be found by calculating $\pi r^2 h$ as the screen shows. A cylinder with radius 3.5 cm and height 18.4 cm has volume approximately 708.11 cm^3.

Notice that $\pi r^2 h$ means $\pi \times r^2 \times h$ on the calculator, as for standard algebraic expressions.

Once values for r and h have been stored in the calculator, they can be used in other formulas too.

Interaction Q

1. Use the values of r = 3.5 cm and h = 18.4 cm to find:
   (i) the area of the top of the cylinder, given by $A = \pi r^2$.
   (ii) the area of the curved surface of the cylinder, $A = 2\pi rh$.
   (iii) the volume of a ball that just fits in the cylinder, $V = 4/3\pi r^3$.

2. A cylindrical jam tin has height 10.5 cm and radius 3.7 cm. Use formulae to determine its volume and the amount of metal used to make it.
3. Measure the height and volume of a soft drink can and use the measurements to determine the capacity of the can in mL. Compare your answer with the quantity shown on the label.
4. Assign a value to M in your calculator. Find the value of each of the following and then explain the similarities and differences between their values:

   $$3M,\ MMM,\ M \times M \times M,\ M^3 \text{ and } M + M + M.$$

   Then repeat by assigning a different value for M and check that the same similarities and differences exist.

   You will do a considerable amount of work with formulae in both our *Measurement* and *Financial Mathematics* books.

Solving equations related to population growth

We have already used some equations earlier in this book. We are now going to look at some equations that are a little more difficult to solve than the linear equations we used. We have chosen to look at equations that are useful when modelling the changes in populations.

The growth of a population, like the population of Australia, can often be modelled with the formula,

$$N = P(1 + r)^x$$

In this formula

N stands for the population size x years from now;
P stands for the population size now;
r stands for the annual rate of growth of the population; and
x stands for a number of years from today.

Like all models, this model depends on some assumptions. The main assumption is that the population growth rate r is the same every year. Any results obtained using this model depend on this assumption. Although we know the assumption is usually not true, it still allows us to get a fairly good approximation of how a population grows in size, especially for a fairly short time period.

Growth rates are often given as percentages, such as $r = 2.5\%$. You need to use a decimal value for r in the formula; in this case $r = 0.025$.

You can tell that this model shows *exponential growth*, since the variable x is an exponent in the formula. You might notice that the formula is similar to that used for compound interest, $FV = PV(1 + r)^N$. The mathematical ideas are the same.

Interaction A

1. Give some reasons why the population growth rate for Australia might not be the same every year.
2. Consult *Yearbook Australia* or some other official source to find the population of Australia and the annual growth rate for last year. (The Internet site of the Australian Bureau of Statistics is a good source. The URL is http://www.abs.gov.au)
3. Use the formula and the growth rate found in question 2 to predict the population of Australia ten years from now.

When formulae are used, it is often necessary to *solve* an *equation* to answer interesting questions.

For example, the population of Australia in 1999 reached 19 million, and the annual growth rate was reported by the Australian Bureau of Statistics to be 1.3%. In what year will Australia's population grow to 25 million?

If you substitute these values into the formula, you get an *equation*:

$$25 = 19 \times 1.013^x$$

To *solve* this equation, you have to find which values of x, if any, make the equation true. In this case, an approximate answer will do, since we know that the model is only an approximation anyway and also an answer to the nearest year or so is probably all we require.

Guess and Check

One way to solve the equation is to try some values for x until the result is close to 25. A quick way of doing this on the calculator is shown in the screen.

The question mark comes from the PRGM (SHIFT VARS) menu.

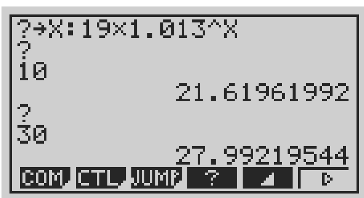
The colon (:) separates two statements and also comes from the PRGM menu. Press F6 and then F5 from this menu.

When you press EXE, the calculator shows a ? so that you can guess a value for x.

After entering the value, press EXE again to check the value of the formula.

The screen here shows that when $x = 10$, the population is about 21.62 (million), which is not enough.

Press EXE again to guess a better value.



The screen shows that $x = 30$ gives a population of about 28 million, which is too large.

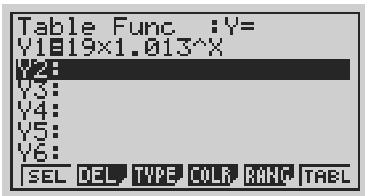
By choosing your next guess carefully after each check, you should be able to get close to a solution (a population close to 25 million) fairly quickly.

Interaction 5

1. Use the Guess and Check method above to find the year in which the population of Australia is predicted to reach 25 million.
2. Suppose that the population growth rate was actually 1.5% per annum. Write the equation you could use to predict when the population will reach 30 million. Solve the equation to find the year.

3. A magazine article, written in 1999, suggested that the population of Australia would reach 50 million by the year 2050.
   (a) Use the formula to write an equation that can be solved to find what constant growth rate would have this effect.
   (b) Solve the equation.

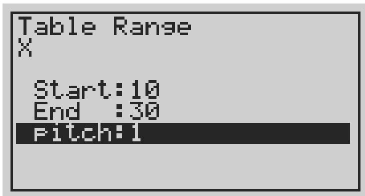
Using a Table



A quicker way of using Guess and Check to solve an equation approximately is to make a table of values. Consider again the equation

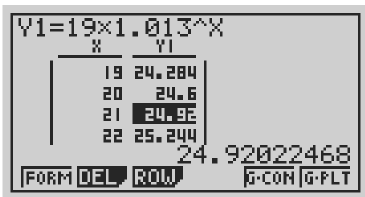
$$25 = 19 \times 1.013^x$$

In TABLE mode, enter 19×1.013^x as Y1.



Use RANG (F5) to select a range of values of x for the table. The screen here shows a starting value of x = 10 and a finishing value of x = 30, going up in steps (pitch) of 1.

Press EXE when all values are entered and construct the table with TABL (F6).

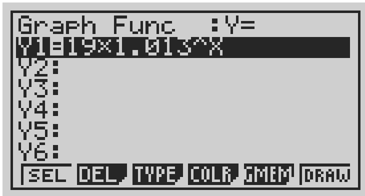


When you scroll down this table using the cursor keys, you can see that after x = 21 years, the population is expected to be just a bit less than 25 (million), while after another year (x = 22), it is a little more than 25 million.

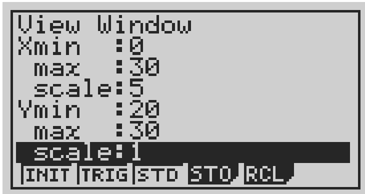
Interaction T

1. Use the Table mode with a Start value of 21 and an End value of 22, with pitch = 0.1 to get closer to an exact solution to the population equation.
2. Write a few sentences to explain why we would be a little uneasy about continuing this sort of process to get more precise solutions.
3. Suppose that the population growth rate was actually 1.4% per annum. Write the equation you could use to predict when the population will reach 30 million. Use a table to solve the equation to find the year.
4. A magazine article, written in 1999, suggested that the population of Australia would reach 50 million by the year 2050.
   (a) Use the formula to write an equation that can be solved to find what constant growth rate would have this effect.
   (b) Use a table to solve the equation.
   (c) Which method do you prefer to use to solve this equation: a table or the Guess and Check method of **Interaction S**? Justify your choice.

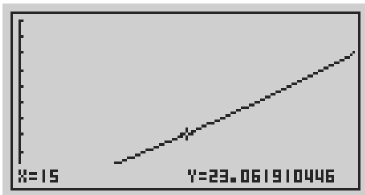
Tracing a Graph



Yet another way of solving the population equation is to draw a graph and then to trace it. In this case, we want to find out when 19×1.013^x has the value 25. So set `Y1` $= 19 \times 1.013^x$ as for the table.



You need to select a viewing window carefully. In this case, we'll consider values from 0 to 30 (years from now) with the population likely to be in the range 20 to 30 (million). The viewing window shown here will be adequate.



Draw the graph, using `DRAW (F6)` and then `TRACE (F1)` it to get a value close to `Y` = 25, representing 25 million population.

The screen here shows that the population after 15 years has only reached about 23 million, so further tracing is needed.

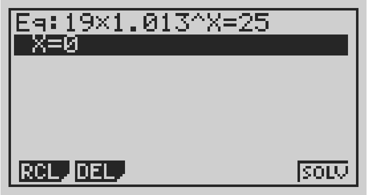
Interaction U

1. Use the `Graph` mode with the view window shown previously to try and get closer to a solution to the population equation. Compare this result with that obtained by the previous two methods.
2. Use a graph to find out when the population is likely to reach 35 million, assuming the growth rate stays constant.
3. The growth rate of Indonesia in 1999 was estimated by the CIA to be 1.46%, with a population of 216 million. At this rate, when would you expect the population to reach 250 million? Use a graph to answer this question.

Using EQUA mode

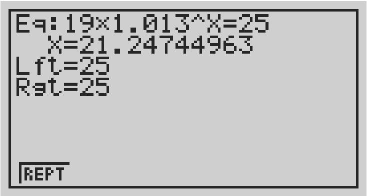
Yet another way of solving the population equation is to use the `EQUA` mode of the calculator. This mode operates as a black box. That is, you give it the equation and it gives you the solution, with little hint of how it achieved the solution. We will again solve the equation $19 \times 1.013^x = 25$.

Enter the `EQUA` mode. Then enter the Solver `(F3)`.



Enter the equation $19 \times 1.013^x = 25$. The exponent can be entered using the hat key (^). The = sign can be accessed by pressing SHIFT and then the decimal point key. Be sure to press EXE to store the equation.

You will note that X=0 appeared on our screen. Your screen may be different. This is the calculator's first approximation (guess) of the solution. It uses a rather sophisticated method of solution but it is similar to guessing and checking in that it iterates until it is happy with the accuracy of the solution. *If you have some idea of the solution and it is more accurate than what the calculator suggests, you should change the* X *value*. If not use the calculator's attempt.



Use SOLV (F6) to solve the equation.

If the Lft and Rgt (standing for left and right) are identical then the calculator has found an accurate solution.

In some cases the calculator will fail to find a solution, and the Lft and Rgt values will differ. You will be instructed to try again. The calculator may ask you to enter an approximation, so it is good to have used one of the methods you explored earlier (if possible) so that you can get an idea of the solution.

Interaction V

1. Use the EQUA mode to solve the equation $1.20t + 0.15 = 5$ which was one of the equations from our mobile phone investigation. Notice that you can enter the equation into the Solver with a T instead of an X, unlike when in GRAPH or TABLE mode
2. Use the EQUA mode to solve the equation $30b + 500 = 50b$, which is the final equation from Tyron and his father's adventures.
3. Solve $p^{20} = 0.1$ with an initial value of $p = 20$. How many times did it take the calculator to find a solution?
4. Use REPT (F1) to solve $p^{20} = 0.1$ again, but this time with the first approximation $p = 100000$ (or 1E5). Follow the directions given by the calculator. Explain how this process differed from that you saw in question 3.
5. You have now used four different methods to find approximate solutions to the population equation. Which one do you prefer and why?

ANSWERS

Some of the questions that have been asked do not have a single correct answer. In such cases, MPA (which stands for many possible answers) will be the answer supplied. In many cases, some supporting comment is supplied.

Interaction A

1. (a) $0.65 \times 12 = \$7.80$

   (b) $\$1.20 \times 6 + 0.15 = \$7.35$
2. $19.50, $18.15
3. $39.33, $36.45

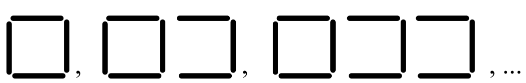
Interaction B

1. X-values don't go up in steps of 1, so that most X-values of interest do not appear.
2. $19.50, $18.15
3. Table will show costs for various times every 15 seconds.
4. 4 mins 30 seconds for Company A, 4 minutes 45 seconds for Company B.
5. Company A, about 3.8 minutes.
   Company B, about 4 minutes.
6. Company A if your calls are kept below 1.5 minutes in duration, otherwise calls of the same duration are cheaper through Company B.

Interaction C

1. Company A: $20 = 1.30t$, so $t \approx 15.4$ minutes.
   Company B: $20 = 1.20t + 0.15$, so $t \approx 16.5$ minutes
2. $0.65
3. $1.35
4. $2.35
5. $1.30t - (1.20t + 0.15) = 0.10t - 0.15$
6. Check with your table
7. Company B is better, except for small values of t.

Interaction D

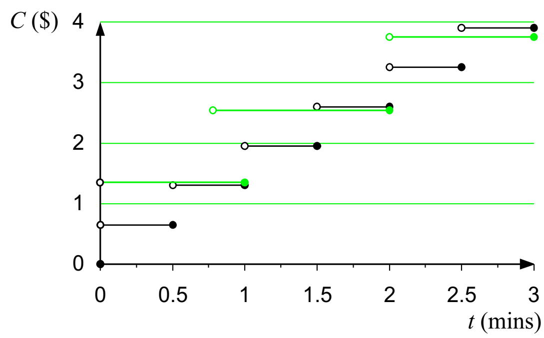
1. 
2. $3 \times 8 + 1 = 25$, which checks.
3. $y = 5x + 2$, so for $x = 200$, $y = 5 \times 200 + 2 = 1002$
4. MPA. With twice as many squares, some people might expect that here should be twice as many toothpicks. But this would count the joining toothpicks twice. In fact, there are $2(3x + 1) - x = 5x + 2$ needed.

Interaction E

1.

t_{min}	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$1\frac{1}{4}$	$1\frac{1}{2}$
C($)	0	0.65	0.65	1.30	1.30	1.95	1.95

t_{min}	$1\frac{3}{4}$	2	$2\frac{1}{4}$	$2\frac{1}{2}$	$2\frac{3}{4}$	3
C($)	2.60	2.60	3.25	3.25	3.90	3.90



3. For Company A, 26 lots of 30 seconds costs $26 \times 0.65 = \$16.90$;

   For Company B, 13 minutes costs $\$0.15 + 13 \times \$1.20 = \$15.75$

Interaction F

1. MPA, eg, graph of `Y4` is not connected (unlike the `G.CON` graph); more points plotted than for `G.PLT`.
2. The graph in `PLOT` mode.
3. MPA, eg, `G.CON` is misleading as it gives the impression that there is a sliding scale between the end of one minute and the start of the next one; in fact there is a 'jump'; `G.PLT` is misleading as it gives the impression that only phone calls of certain lengths have an associated charge.
4. MPA, eg `Y1` has a sliding scale, so that phone calls of length 1 minute and 10 seconds cost less than those of 1 minute and 20 seconds (when in fact each costs the same).
5. MPA, eg, `Y5` is shown with a step graph, while `Y2` is shown as a stright line.

Interaction G

1. MPA, eg., the graph for Company A has twice as many 'steps' as that for Company B, since units of 30 seconds are charged for instead of units of one minute.
2. Company A: $19.50, Company B: $18.15
3. Company A: $\leqslant$4 1/2 minutes, Company B: $\leqslant$4 minutes
4. Company A: $\leqslant$3 1/2 minutes, Company B: $\leqslant$4 minutes
5. MPA, for most calls, Company B is better; however, sometimes Company A is better (e.g. question 3).

Interaction H

1. MPA, eg, If $P + 2 = 5$, then P must be 3, which also fits with $3P = 9$. Then if $P + Q = 17$ and $P = 3$, Q must be 14. Store the values 3 and 14 for P and Q, then clear the screen and repeat the calculations shown.

2. $A = 4$, $B = 6$; $B = 6$ and is 2 more than A, so $A = 4$.

   $M = 5$ and $N = 5$; $M - N = 0$, so M and N must be the same. $M + N = 10$, so each must be 5.

   $J = 4$, $K = 1$; $2J + K$ is J bigger than $J + K$ and 9 is 4 bigger than 5, so $J = 4$.

   $X = 4$, $Y = -1$; $X + Y$ is $2Y$ bigger than $X - Y$, so $2Y = -2$ and so $Y = -1$. Then $X = 4$.

3. MPA

Interaction I

1. MPA. One possibility is to choose the `INIT` view window, but there are many others such as $-2.7 < x < 9.9$ and $-0.7 < y < 5.5$. In each case, the graph should be a line, sloping upwards from left to right at an angle of 45 degrees and intersecting the axes at $(-1,0)$ and $(0,1)$.

2. MPA. Eg, one choice to produce a graph that slopes *less* steeply is the STD view window, for which each axis goes from -10 to 10. One choice that produces a graph that slopes *more* steeply is $-1 < y < 1$ and $-6.3 < x < 6.3$.

3. With the `INIT` view window, the two lines are perpendicular to each other; with the other window, this is not the case.

4. In each case, the graph is a line, sloping upwards left to right and intersecting the axes at $(0,-1)$ and $(1,0)$. With the `INIT` view window, the line is at 45 degrees to the horizontal, while it is less steeply sloping for the other window.

5. With the `INIT` window, the tracing goes up in steps of 0.1 on the x-axis and the point $(1.5,2)$ is traced to directly. With the other window, the tracing goes up in less convenient steps (of $10/26 \approx 0.07936507937$) and the point $(1.5,2)$ cannot be traced to directly.

6. Only (a) and (c) have the desired effect. (Each is a multiple of the values from the INIT view window). There are (infinitely) many other possibilities. Eg., add the same amount to each of the INIT x-values to get $-4.3 < x < 8.3$ instead of $-6.3 < x < 6.3$, leaving the y-values the same.

Interaction J

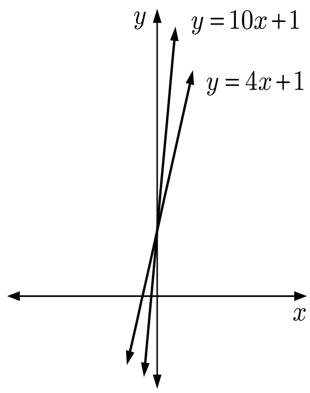
1. For each function, $b = 1$ and m is the multiplier before the x.

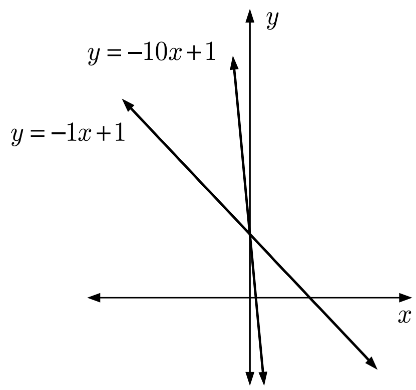
2. The graph become steeper.

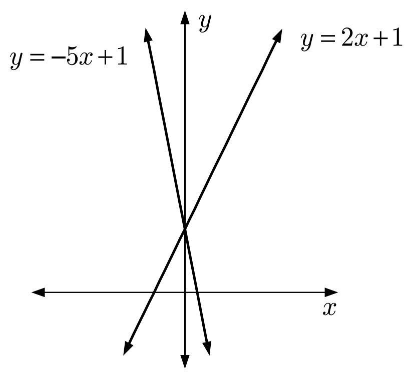
3. The graph will be a reflection about the line $x = 1$; The graph is a line sloping downwards from left to right, rather than upwards from left to right.

4. When the rate of change is zero, there is no change. So all the x values are the same for all x values.

Interaction K

1. 

2. 

3. 

4. MPA, eg the angles don't change by the same amount each time, even though the A values do change by a constant amount.

Interaction L

2. When b is positive, the graph cuts the y axis above the x axis (i.e., $y > 0$); when b is negative, the graph cuts the y axis below the x axis (i.e., $y < 0$).

3. The slope of the graph does not change, but as b changes from 4 to 10, the point where the graph cuts the y axis moves up.

4. Slope does not change, but as b goes from -4 to -10, the point where the graph cuts the y axis moves down.

Interaction M

1. (i) 100 (ii) 50 (iii) 17
2. $530
3. $30 or less
4. 1000

Interaction N

1. For $x = 3$, each expression has a value of 68. For $x = 5$, each has a value of 156 and for $x = -2$, each has a value of -12. It is much quicker to use the short expression.
2. $-4a^2 + 11a - 3$
3. The expressions are *not* equivalent; Change the second one to $2t^2 - 5t + 17$.

Interaction O

1. $(x + 3)^2 \neq x^2 + 9$ as the tabled values are generally different (except for the case $x = 0$).
2. Jarryd was corect. $5x(2x + 1) = 5x \times 2x + 5x \times 1 = 10x^2 + 2x$. Meg didn't multiply $5x$ by 1, while Peter multiplied $2x$ by 1 instead of adding.
3. $(x - 5)^2 = x^2 - 10x + 25$

   $(x - 7)^2 = x^2 - 14x + 49$

   $(x - a)^2 = x^2 - 2ax + a^2$
4. Kaye was incorrect. $3x^2 \times 5x = (3 \times 5)(x^2 \times x) = 15x^3$.
5. All are described as correct by the calculator, although only the first one is generally true. $x = 0$ is a special case, for which the given equalities are true, although not generally true.

Interaction P

1. $(x + 1)^2 \neq x^2 + 1$. Generally, the graphs of $y = (x + 1)^2$ and of $y = x^2 + 1$ are different, intersecting only at $x = 0$.
2. $(x + 5)^2 = x^2 + 10x + 25$

   $(x + 7)^2 = x^2 + 14x + 49$
3. $x^2 + 2x + 7$ is 3 more than $x^2 + 2x + 4$, so must be $(x + 1)^2 + 3 + 3$. That is, $x^2 + 2x + 7 = (x + 1)^2 + 6$.
4. The graphs of $y = x(x + 1) - x(x - 1)$ and $y = 2x$ are the same.

   $x(x + 1) - x(x - 1) = x^2 + x - x^2 + x = 2x$.
5. No. The graph of $y = x^2 + 4$ does not show on the INIT screen. To see this, draw the graph and then press the up arrow key (twice).

Interaction Q

1. (i) 38.48 cm^2 (ii) 404.64 cm^2
   (iii) 179.59 cm^3
2. Volume is 451.59 cm^2; the amount of metal is 330.12 cm^2.
3. MPA. Our can was 12.5 cm high and 6.5 cm in diameter, but was not precisely a cylinder. A cylinder with these measurements would hold 415 cm^2, or 415 mL, but the can label showed only 375 mL. The can is not filled completely.
4. $3M = M + M + M$ and $M^3 = MMM = M \times M \times M$.

Interaction R

1. MPA, eg changing preferences for family sizes, migration trends, improved health practices leading to lower death rates, etc.
2. For 1999, population was 19 million and growth rate was 1.3%.
3. MPA. Using data for question 2, $P = 19(1.013)^{10} \approx 21.6$ million people.

Interaction S

1. 21 years after 1999; i.e. near the year 2020.
2. $19 \times 1.015^x = 30$ gives $x \approx 30.7$, so near the year 2030.
3. (a) $19 \times (1 + x)^{51} = 50$.

   (b) $x = 0.019$, so the growth rate is 1.9%.

Interaction T

1. x is closer to 21.2 than 21.3.
2. It's not sensible to get an answer that is too precise as the growth rate is unlikely to be constant anyway.
3. $19 \times 1.014^x = 30$ gives $x \approx 33$, so the year is around 2032.
4. (a) $19 \times (1 + x)^{51} = 50$

   (b) $x = 0.019$, so the growth rate is 1.9%.

   (c) MPA. Some people find the table method more efficient.

Interaction U

1. The closest value is $x \approx 21.19$. This result is similar to the previous result.
2. After about 47 years, or around 2046.
3. Draw the graph of $y = 216 \text{ x } 1.0146^x$ and trace to find where $y \approx 250$ and $x \approx 10$, so the year is about 2009.

Interaction V

1. $t \approx 4.04$
2. $b = 25$
3. $x \approx 0.891$, which takes two attempts, since the initial value is much larger than the actual solution.
4. The initial value is so far away from a solution that the calculator suggests that you adjust the initial value and try again.
5. MPA

NOTES