NSW GENERAL MATHEMATICS Preliminary Course

Mathematical Interactions

Measurement

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Mathematical Interactions: Measurement

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About this book

Calculators are too often regarded as devices to produce answers to numerical questions. However, a graphics calculator like the Casio CFX-9850GB PLUS is much more than a tool for producing answers. It is a tool for exploring mathematical ideas, and we have written this book to offer some suggestions of how to make good use of it when exploring ideas related to measurement.

We assume that you will read this book with the calculator by your side, and use it as you read. Unlike some mathematics books, in which there are many exercises of various kinds to complete, this one contains only a few 'interactions' and even less 'investigations'. The learning journey that we have in mind for this book assumes that you will complete *all* the interactions, rather than only some. The investigations will give you a chance to do some exploring of your own.

We also assume that you will work through this book with a companion: someone to compare your observations and thoughts with; someone to help you if you get stuck; someone to talk to about your mathematical journey. Learning mathematics is not meant to be a lonely affair; we expect you to interact with mathematics, your calculator and other people on your journey.

Throughout the book, there are some calculator instructions, written in a different font (like this). These will help you to get started, but we do not regard them as a complete manual, and expect that you will already be a little familiar with the calculator and will also use our *Getting Started* book, the *User's Guide* and other sources to develop your calculator skills.

Measurement is one of the topics in General Mathematics, mainly because it is a fundamental idea in mathematics and in the applications of mathematics to the real world. Many practical uses of mathematics begin with some measurements of quantities such as length, time and angle and these measurements are used in turn to derive other measures such as those for area, speed and growth rate. You will learn to use measurement formulae in various ways to deal with practical questions, and will find your calculator of value in dealing with the relevant computations. Both tables and graphs will help you to deal with practical situations involving measurement. Calculator programs are useful for evaluating measurement formulae that will also be used.

Since no measurements in the real world are exact, an important aspect of measurement concerns the accuracy of information and the consequences of any measurement errors involved. You will find that your graphics calculator is especially useful in helping you to understand and to explore the effects of measurement errors and to make sensible decisions about accuracy. Although we have sampled some of the possible ways of using a graphics calculator to learn about these topics, we have certainly not dealt with all of them.

We are grateful to Ian Freney, for his careful reading and useful feedback to earlier drafts of this book, and to Anthony Boys, for his effort in editing and checking the solutions.

We hope that you enjoy your journey!

Barry Kissane
Anthony Harradine

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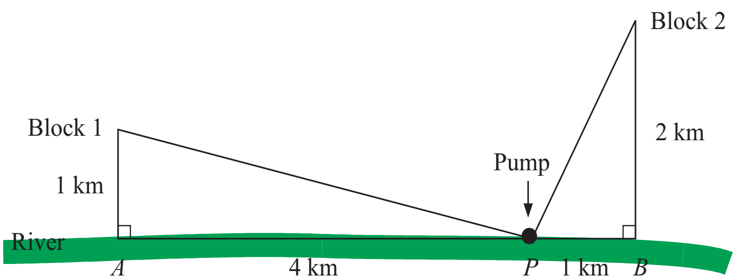
Using the Theorem of Pythagoras to find the best position

Measuring things is a fundamental part of life. Even more so is the ability to determine the best measure for a given situation. The best measure often results in better performance or saving some money.

A orange farmer named Chloe owned two fruit blocks that she wanted to irrigate using water from the nearby river. She worked out that it was possible to put one pump on the bank of the river and that it could service both blocks rather than having to use two pumps. Two pipelines would run from the pump, one to each block. A plan of what she did can be seen below.



Interaction A

1. Use Pythagoras' Theorem to work out the length of the pipe required to take water from the pump to block 1.
2. Use Pythagoras' Theorem to work out the length of the pipe required to take water from the pump to block 2.
3. Hence state the total length of pipe that was used.

4. Had the pump (P) been placed at a different position on the river, would the same amount of pipe have been required? Re-draw the diagram with the pump in a position of your choice and re-calculate the total length of pipe required. Do this for two different positions and then tabulate your results along with those of your classmates in a table with the following headings: *distance* AP, *distance* PB, *length of pipe from block 1 to pump*, *length of pipe from block 2 to pump*, *total length of pipe required*.
5. Study your table carefully. Is there a *best* place for the pump to be positioned? If so, describe where the pump should be placed.
6. If the pump was placed x kilometres along the river from A, how far would it be from B?

From **Interaction A** you should be convinced that the total length of pipe required varies and is dependent on where the pump is positioned on the river. It is unlikely though that you found the optimal (ie. best possible) position. To do this you would have to try many many cases, not just the 20 or so that you have from your class. You should, however have some idea about where the optimal position is.

Using the calculator we can calculate and display many cases very quickly. First we need to develop a function for the quantity we want to calculate and display.

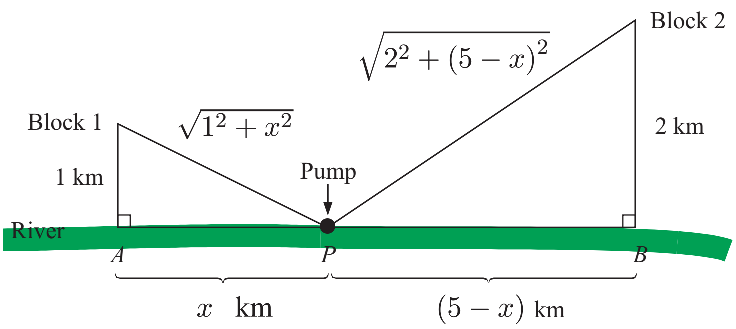
By letting the pump be placed x kilometres from A (which means it is $(5 - x)$ kilometres from B) and by using the Theorem of Pythagoras we can build some functions that determine the length of the two runs of pipe that are required for all positions that the pump can take:

$$\text{distance from block 1 to P} = \sqrt{1^2 + x^2}$$

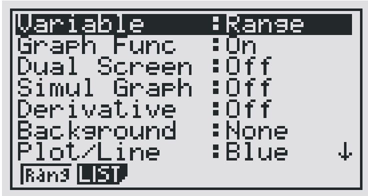
$$\text{distance from block 2 to P} = \sqrt{2^2 + (5-x)^2}\text{ , and so}$$

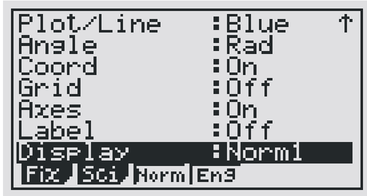
$$\text{total length of pipe} = \sqrt{1^2 + x^2} + \sqrt{2^2 + (5-x)^2}$$

These are illustrated on the diagram below.

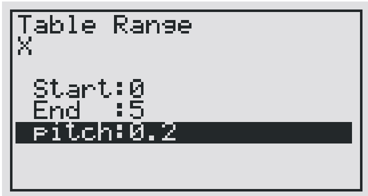


Enter TABLE mode and use SET UP (SHIFT MENU) to ensure that the settings are as shown on the next page.

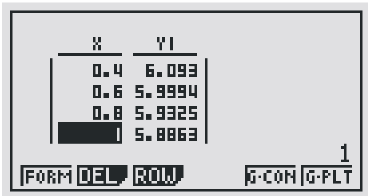




Then enter the function for the total length of pipe. It will run over the screen a little. Be careful with your use of brackets when entering the function.



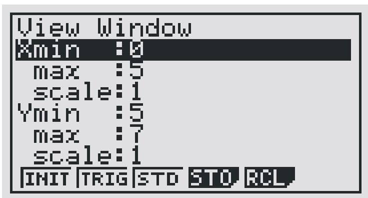
Use RANG (F5) to set the values for the distance AP you want the calculator to use to calculate and display the total length of pipe required. We have chosen to start with x at zero kilometres and increase its value by 200 m (pitch of 0.2) at a time up to five kilometres. Zero and five are the obvious limits for the value of x. Press EXIT.



Use TABLE (F6) to produce the table of values. Part of our table is shown here.

Study this table carefully and see if it reveals the optimal values.

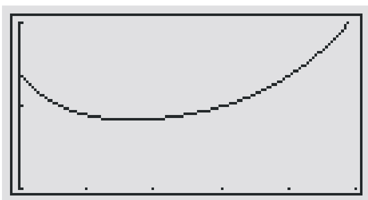
In addition to studying a table of values it is often helpful to draw a graph of them. This may help us to see the optimal values more easily. First we must tell the calculator what limits and scales to use on the axes of the graph.

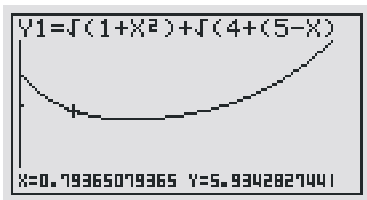


Use V.WIN (SHIFT F3) to do this. Set the values requested as shown opposite. Note that the Ymin and max have been chosen by considering the values produced in the table. The scale value simply sets where marks on the axes of the graph are to appear. Press EXIT.

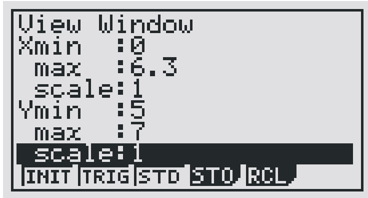
Use G·CON to produce a plot of the total length of pipe required by distance AP. Your graph may be a little different to ours depending on the values you set for the table range. Why is G·CON suitable in this case?

Then use TRCE (SHIFT F1) to trace along the graph and see if you can locate the optimal point.

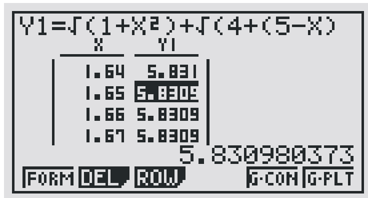




Notice that the values of the coordinates that are achieved from tracing the function are rather horrid. This is due to the number of pixels that form the calculator's screen. In this case, x-values from 0 to 6.3 work nicely. Change the viewing window parameters as shown here and trace the graph again.



You should have been able to see from the table and the graph that the optimal point to place the pump is between 1.4 and 1.8 kilometres from A. So, now use RANG to set the table range as shown below. The table you produce will allow you to *home* in on a more accurate approximation for the optimal value of AP. Note that you need to have the cursor on the value you are interested in to see its value to more than four decimal places.

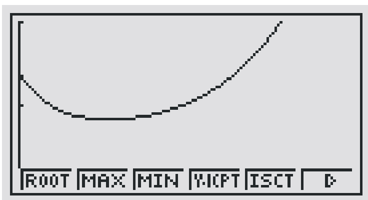


You can now change the table range again and continue to improve the accuracy of the approximation of the optimal value for AP. Do you think you will be able to find the exact value using the calculator? Why or why not?

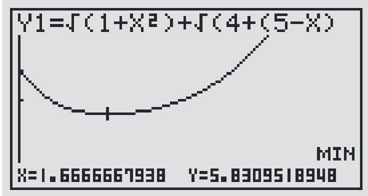
Interaction B

1. Find the value of AP for the optimal position of the pump (accurate to 4 decimal places) and the corresponding length of pipe, accurate to 9 decimal places.
2. Consider the accuracy of 4 decimal places. You should have found that if AP is 1.6666 or 1.6667 km (that is 166670 centimetres) the length of the pipe required is 5.830951895 kilometres, that is 583095.1895 centimetres. Do we need to be so accurate in this problem? Do we ever need to be this accurate?
3. If the price of the pipe used by Chloe was \$15 per metre, calculate how much she could have saved, compared to what she actually did save (see the original plan), had she known the optimal placement for the pump.

The calculator can give you a very good approximation to the minimum length of pipe and corresponding distance for AP without all of the detective work we have done.



Enter GRPH mode, and you will see the function we entered in TABLE mode is present. Use DRAW (F6) to draw the graph of this function.



Access G–SLV (SHIFT then F5) and then use MIN (F3) to make the calculator locate the minimum point on the graph. Watch as the little cursor travels along the graph and eventually stops at the minimum point.

Investigation:

Once Chloe had heard about our analysis she thought about her mathematics education. She went away and did some analysis of her own and claimed that the distance AP, for the optimal position of the pump was exactly $1\frac{2}{3}$ kilometres. When asked how she calculated it she said, “I simply used a little bit of geometry which I remembered learning at school. It involved some similarity and the fact that the shortest distance between two points is along a straight line between the two points.”

Determine how Chloe arrived at $1\frac{2}{3}$ kilometres.

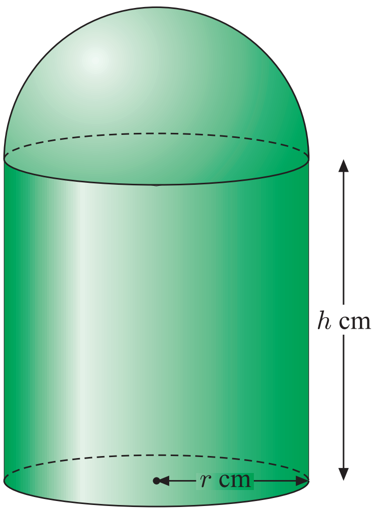
Making a better motor mower engine

Simple measurement formulae such as those for area and volume can be used to make mathematical models of quite complex, practical situations. We do, however, need to be careful that the model formed makes sense in the real world.

This section uses volume and area formulae. Where algebraic manipulation of these formulae is required, we have included all steps. We do not expect you to be able to complete the steps for yourself.

Motor mowers have engines with a capacity of 100 cc (ie. 100 cubic centimetres). The internal structure of the engine is essentially a cylinder (in which a piston moves up and down) with a hemispherical combustion chamber on top of it. A new 100 cc engine was to be designed by a manufacturer of motor mowers.

The combined volume of the cylinder and hemisphere is 100 cc. The manufacturers wanted to know the dimensions of the cylinder (height and radius) that would result in the volume being 100 cc.



This diagram illustrates the piston cylinder and combustion chamber. The two are bolted together with a seal in between them so that no gases leak out.

Interaction C

1. Is there only one combination of radius and height values that will result in this structure having a volume of 100 cc?
2. Use the formula for the volume of a cylinder and a hemisphere to show that each of the following combinations will result in a combined volume of 100 cc (or just about!)

r = 1.400 cm	and	h = 15.307 cm
r = 2.400 cm	and	h = 3.926 cm
r = 3.400 cm	and	h = 0.487 cm

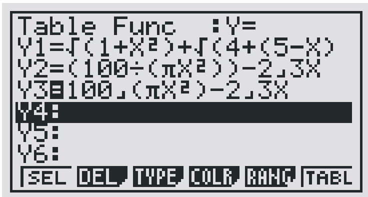
3. A simple calculator *program* can be used to take the repetition out of calculations such as those you did in the previous part of this interaction. Enter RUN mode, access the PRGM menu by pressing SHIFT then VARS. Enter the program seen on the following screen. The arrow is available on the key pad, the letters result from the use of the red ALPHA key while the ? and colon are available from the PRGM menu (seen at the base of the screens below). Pressing EXE results in the calculator asking for the value of R. Enter 1.400 and press EXE. The value of H is then requested, enter 15.307 and press EXE. The volume of the structure results as shown below. Pressing EXE starts the program again. Check your answers to part 2.

4. Draw a scale diagram that illustrates what each of the structures, with dimensions used in part 2, would look like. Can you draw a diagram as accurate as those values given? If not, be accurate to the nearest millimetre.
5. Describe what appears to happen to the height as the radius gets larger.
6. Clearly many different configurations would be possible. Does any one of the possible structures seem better than another?

We can use the calculator and the formulae for the volume of a cylinder and a hemisphere to display many combinations of radius and height that will result in a combined volume of 100 cc. Consider the following; it is quite complex algebra but possible to follow:

$$
\begin{aligned}
V_{cylinder} + V_{hemisphere} &= V_{cylinder\,\&\,hemisphere} \\
\pi r^2 h + \tfrac{1}{2}\left(\tfrac{4}{3}\pi r^3\right) &= V_{cylinder\,\&\,hemisphere} \\
\pi r^2\left(h + \tfrac{2}{3}r\right) &= 100 \\
h + \tfrac{2}{3}r &= \frac{100}{\pi r^2} \\
h &= \frac{100}{\pi r^2} - \tfrac{2}{3}r
\end{aligned}
$$

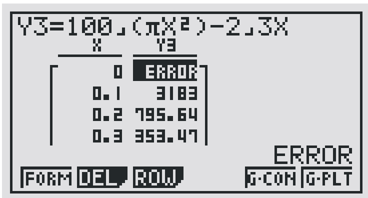
We now have a function for the height of the structure in terms of its radius. Hence if we let r have a certain value, like 1.4 cm, we can calculate the corresponding height that is required to give a combined volume of 100 cc.

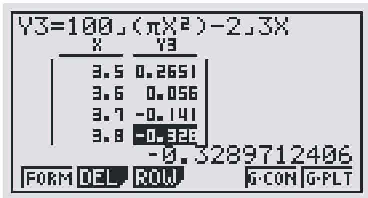


In TABLE mode enter the function $\frac{100}{\pi r^2} - \frac{2}{3}r$. We have entered it in two slightly different ways, one using the fraction key instead of the divide key. Note that r is replaced by X as it is the variable the calculator recognises in TABLE mode. (It is on the X,θ,T key).

The radius must be positive and it seems unlikely that it can be more than 5. To consider many possibilities, we will make a table of values with the radius increasing by 0.1 each time. To do this, set the table range to have a start value of 0, an end of 5 and a pitch of 0.1. Use TABL (F6) to produce the table of radii and heights. Study it carefully, using your arrow keys to navigate it.

Notice that as the radius gets larger, the height gets smaller. If you think about the physical structure of the cylinder and combustion chamber, this should seem reasonable.





Interaction D

1. Why does the calculator return ERROR when the radius value is 0 cm?
2. Why was a negative height value returned when the radius was 3.8 cm and beyond? Would a negative height make any sense in the design of a motor?
3. Change the pitch of the table range to 0.05. Study the table. Can you decide on the optimum combination of radius and height values?

One of the people involved in the building of this engine had an idea. Since they had a choice of dimensions he thought, to save as much money as possible, they would choose the dimensions that gave the *minimum* surface area. This would mean that they would use the least possible amount of material in the construction.

Note that the cylinder has an open top and bottom. Hence the surface area of the combined structure would be:

$$\begin{aligned} SA_{cylinder\,\&\,hemisphere} &= SA_{cylinder} + SA_{hemisphere} \\ SA_{cylinder\,\&\,hemisphere} &= 2\pi rh + \tfrac{1}{2}\left(4\pi r^2\right) \\ SA_{cylinder\,\&\,hemisphere} &= 2\pi rh + 2\pi r^2 \end{aligned}$$

Interaction E

1. Calculate the surface area of the complete structure for the following dimensions:

r = 1.400 cm	and	h = 15.307 cm
r = 2.400 cm	and	h = 3.926 cm
r = 3.400 cm	and	h = 0.487 cm

2. Enter the program seen on the screen below in RUN mode and use it to check your answers to part 1.

3. Does there seem to be any pattern to how the surface area varies as the radius values get larger? If you think you need to try a few more cases to answer this question, then do so.

Let us further develop our function for $SA_{cylinder\,\&\,hemisphere}$ so we have only one variable, the radius, that needs to be substituted for. Using $h = \dfrac{100}{\pi r^2} - \frac{2}{3}r$,

$$\begin{aligned} SA_{cylinder\,\&\,hemisphere} &= 2\pi rh + 2\pi r^2 \\ SA_{cylinder\,\&\,hemisphere} &= 2\pi r\,(h + r) \\ SA_{cylinder\,\&\,hemisphere} &= 2\pi r\left(\frac{100}{\pi r^2} - \tfrac{2}{3}r + r\right) \end{aligned}$$

Interaction F

1. Show that $SA_{cylinder\,\&\,hemisphere} = 2\pi r\left(\frac{100}{\pi r^2} - \frac{2}{3}r + r\right)$

   can be further refined to $SA_{cylinder\,\&\,hemisphere} = \frac{2}{3r}\left(300 + \pi r^3\right)$.

   Continue on with the rest of this interaction even if you can not do this part.

2. Enter the function $\frac{2}{3r}\left(300 + \pi r^3\right)$ into TABLE mode of the calculator. You will need to use X in place of r. Generate values for this function for radius values from 0 to 3.8 cm at intervals (pitch) of 0.1 cm. Why has a maximum value of 3.8 cm been suggested? Find the radius value in this table that results in the structure with the minimum surface area.
3. Modify the range values of the table to find the radius, correct to 2 decimal places that will result in the minimum surface area.
4. Determine the height value that corresponds to the radius value that you found in part 3. Draw a scale diagram of the engine for this case.
5. Are you surprised by your answer to part 3? Explain why you have received such a *strange* value.

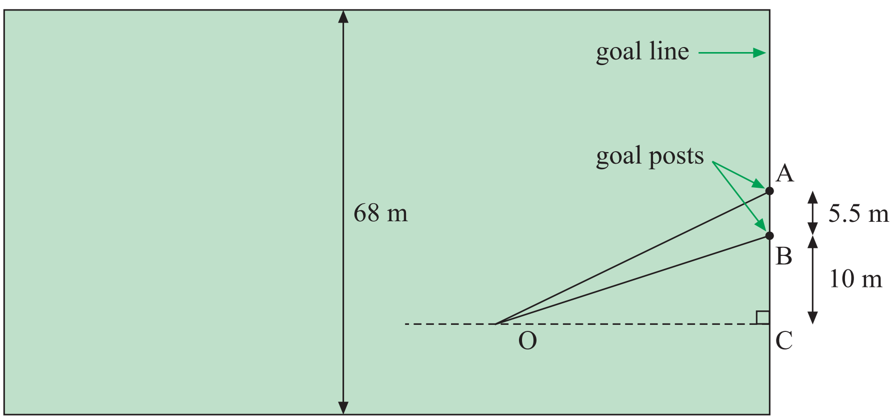
Maximising a sporting chance using trigonometry

If triangles are involved in a problem, trigonometry may help us to find a solution. This example uses trigonometry to solve a sporting problem.

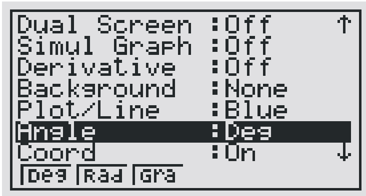
When a Rugby League player takes a kick at goal after a try has been scored, he can take the kick along a line perpendicular to the goal line as far as he likes from the goal line. Just how far from the goal should he take the kick? Is one distance any better than another?

First let us look at a plan of the situation facing the kicker after a team mate has made a try 10 metres from the right goal post.

Let A and B be the goal posts (which are placed 5.5 metres apart), O be the point from which the kicker chooses to kick and C be the point where the try was made.



The *room for error* that the kicker has could be thought of as the size of angle AOB. The larger the angle, the better the situation as far as the kicker is concerned. The question is – how does this angle change as distance OC (the distance chosen by the kicker) changes?



Before starting **Interaction G**, enter TABLE mode and use SET UP (SHIFT then MENU) to set the working unit of an angle to be degrees. You will need to use the down arrow to find the Angle option.

Interaction G

1. Imagine that the kicker chooses to go back 5 metres from C along OC. Draw a diagram to represent this situation. Find the measure of angle AOC and BOC using your knowledge of the tangent function (tan) and hence find the size of the angle AOB (the kicker's room for error). Quote answers correct to 2 decimal places.
2. Repeat part 1, but for the case where the kicker chooses to go back 10 metres.
3. Repeat part 1, but for the case where the kicker chooses to go back 15 metres.
4. Repeat part 1, but for the case where the kicker chooses to go back 20 metres.
5. Repeat part 1, but for the case where the kicker chooses to go back 25 metres.
6. Tabulate your solutions to parts 1 to 5. Describe what appears to happen to the kicker's room for error as the distance that the kick is taken from the goal line increases.

We can use the calculator to calculate and display many cases rather than just the five you dealt with in **Interaction G**. First we must formulate a function that will generate the value of angle AOB for *any* value of distance OC. If we let distance OC be x metres, then

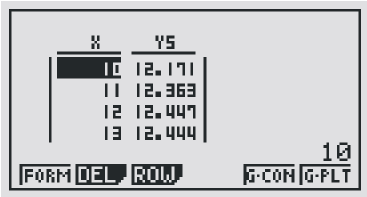
$$\begin{aligned} \text{angle AOB} &= \text{angle AOC} - \text{angle BOC} \\ \text{angle AOB} &= \tan^{-1}\left(\frac{15.5}{x}\right) - \tan^{-1}\left(\frac{10}{x}\right) \end{aligned}$$

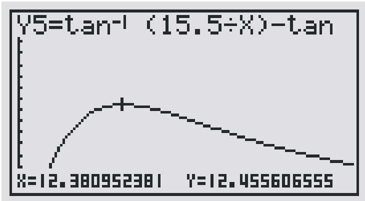
Enter TABLE mode of the calculator and enter the function

$$\tan^{-1}\left(\frac{15.5}{x}\right) - \tan^{-1}\left(\frac{10}{x}\right).$$

Produce a table of values (after setting the appropriate range for x). Study this table carefully and determine, correct to the nearest metre, the distance from C, along OC, for which the kicker will have the greatest room for error.

Our table and graph below give an indication of the required result. You should determine the value correct to 1 decimal place to be sure of your *nearest metre* answer.



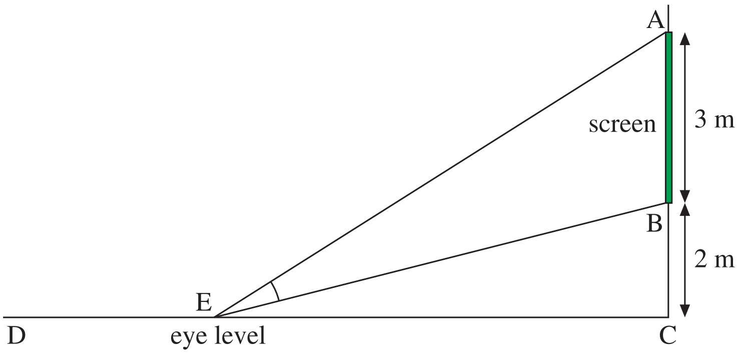


Interaction H

1. Does your result, of where the kicker should take the kick from, seem to agree with the position that the professionals choose to kick from? If not explain why not – are there any other factors that should be taken into account rather than just angle?
2. For the situation when a try is made 10 metres from the right goal post, find the kicker's room for error if the kicker goes back 10 metres compared to the angle if he goes back 16 metres. Do you think the difference between these angles would be of concern to the kicker? Explain why.
3. Determine, to the nearest metre, where the kicker should take his kick at goal if the try is made 15 metres from a goal post and he wants the greatest room for error.
4. Determine, to the nearest metre, where the kicker should take his kick at goal if the try is made 20 metres from a goal post and he wants the greatest room for error.
5. Determine, to the nearest metre, where the kicker should take his kick at goal if the try is made at the furthermost point from the goal post and he wants the greatest room for error. Consult the diagram presented earlier for measurements.

Investigation:

An analysis similar to that used for the Rugby problem could be applied in order to solve the problem of maximising one's viewing angle in a picture theatre. Consider the following plan of a picture theatre with a horizontal floor.



Determine where a person should sit in this picture theatre in order to maximise their viewing angle.

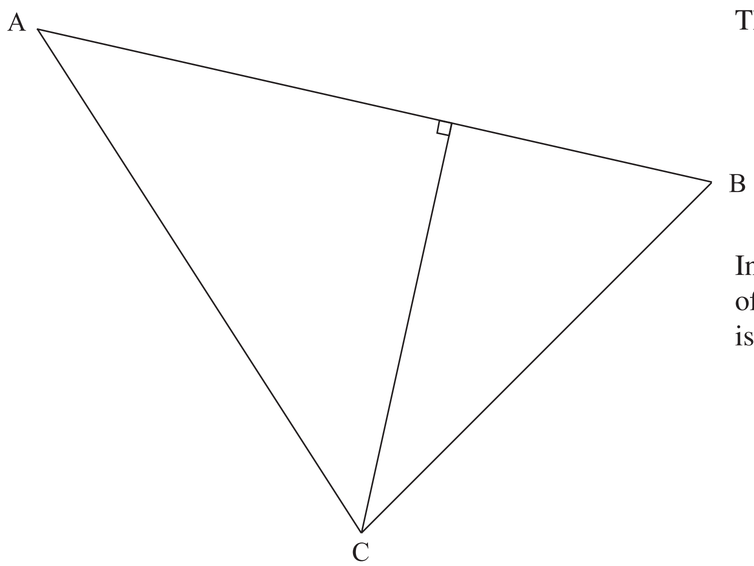
Also investigate the situation of this theatre having an inclined floor. Determine where one should sit for a variety of different inclinations.

Measuring areas, volumes and heights accurately

So far in this book, we have used measurement relationships to construct and use mathematical models. In order to interpret your mathematical model realistically, you need to take account of the errors of any measurement involved. When physically making measurements, you need to pay careful attention to the accuracy of the particular measuring device. These issues are the focus of the rest of this book.

People have been finding the area of plane figures since ancient times. For example, the ancient Greeks and Egyptians had good methods of finding areas of triangles, which we still use today. All shapes with straight sides can be divided up into triangles (a process we call *triangulation*). You can use this process to find the area of any polygonal shape.

One way of finding the area of a triangle uses two measurements: the length of one side and of a perpendicular height.



The area can be found using the formula:

$$Area = \frac{base \times height}{2}.$$

In the triangle ABC, shown here, only one of the three possible perpendicular heights is shown.

Interaction 1

1. Use a ruler to measure the side lengths and perpendicular height of the triangle ABC from C. Do not quote the measurements more accurately than the ruler can measure them. Then use the formula $Area = \frac{base \times height}{2}$ to find the area of the triangle with AB regarded as the base.
2. Use a ruler and protractor or a mathematical template to draw the other perpendicular heights of the triangle ABC. Measure carefully all three sides and perpendicular heights.
3. Use the formula $Area = \frac{base \times height}{2}$ to find the area of the triangle using BC as the base and then using CA as the base.
4. Compare your answers with some other students. What is the difference between the smallest and largest estimates of the areas of the triangle? Draw a square, the area of which equals this difference. Decide what is the best value to give for the area of the triangle.
5. Explain why people get different results from each other and different results with each formula.

Even when you measure carefully, *errors of measurement* are inevitable.

For example, the side AB of the triangle above is 9.2 cm long, to the nearest millimetre. You can't get closer than this to the actual length, unless you have a ruler that measures more accurately than millimetres.

In any case, the thickness of the lines (and so the size of the 'points' A and B) won't allow you to get closer than this. Look at your protractor carefully and you will see that the same is true for measuring angles with a protractor.

For this reason, we think of 9.2 cm, measured to the nearest millimetre, as a measurement somewhere between 9.15 cm and 9.25 cm.

Another way of thinking about this is to say that the length of AB is 9.2 ± 0.05 cm.

As seen earlier it is often useful to use a short calculator program for evaluating a formula, especially when you want to use the same formula several times. Here is another example. An Egyptian who lived in the first century AD, Heron of Alexandria, was responsible for a formula for finding the area of a triangle when the lengths of the sides are known.

If the sides of the triangle are represented by a, b and c, then Heron's formula can be written as:

$$\text{Area} = \sqrt{S\,(S-a)\,(S-b)\,(S-c)}$$

where S represents half the perimeter of the triangle, $S = \frac{a+b+c}{2}$.

The screen shows a program called *HERON* for finding the area of a triangle using Heron's Formula. The program first asks for the lengths of each of the three sides, a, b and c. It then calculates S according to the definition above. Finally it calculates the area.

You may find it useful to refer to our *Getting Started* book for advice about entering programs into your calculator.

First enter Program mode with MENU B and press NEW (F3). Name the program and enter the program steps, pressing EXE after each line.

The question mark command is available by activating the PRGM menu (shown on the previous screen) by pressing SHIFT then VARS. The ↲ symbol shows where EXE was pressed. Press EXIT (twice) when finished entering the program to return to the program list.

When you start the program, the calculator displays a question mark to request the values for a, b and c.

It then calculates and displays the area. The example here shows that the area of a 3-4-5 triangle is 6 square units.

Press EXE to restart *HERON* and then enter data for another triangle.

Interaction J

1. Make sure that *HERON* works by using it to check that the area of a 5-12-13 triangle is 30 square units.
2. Using the measurements of the side lengths of triangle ABC (to the nearest millimetre) and *HERON,* find the area of triangle ABC. Compare your answer with the answers from **Interaction I**. Can you explain any differences in the results?
3. Suppose that each of the *actual* side lengths was half a millimetre *less* than the values you measured. What would be the area of the triangle for this case?
4. Suppose that each of the *actual* side lengths was half a millimetre *more* than the values you measured. What would be the area of the triangle for this case?

The example of the triangle shows that, with careful measurement, a fairly accurate result can be found, but not an *exact* result. An exact result comes from an error free process.

Sometimes it's hard to measure things with a high degree of accuracy and, even if you do, the effects of small errors can be rather large.

Jillian wanted to find out the volume of air in her soccer ball, which was already inflated. She knew that the soccer ball was not quite a sphere, since it was covered in (slightly curved) regular pentagons and hexagons. But she thought that a sphere would be a good enough model, especially when the ball was inflated.

Jillian knew that the formula for the volume of a sphere was

$$V = \frac{4}{3}\pi r^3$$

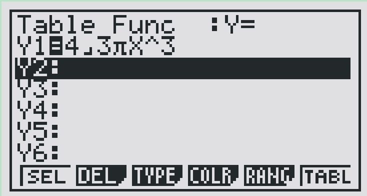
where V stands for volume and r for radius.

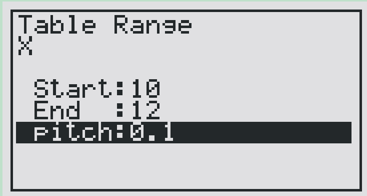
But it was not easy to measure the radius of the ball.

She tried a few methods and finally decided that it was about 11 cm in radius, but felt that her measurement could have been out by as much as a centimetre in either direction.

Interaction K

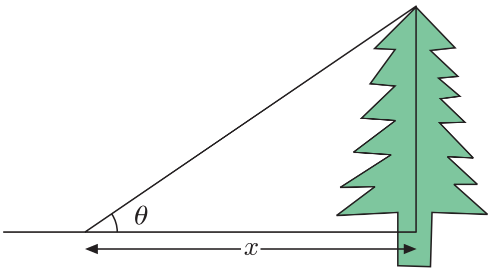
1. What is the volume of the soccer ball, if the radius is exactly 11 cm?
2. Assuming that the radius of the ball is most accurately described as 11 ± 1 cm, use TABLE mode of your calculator to explore the range of possible values for the volume, as shown in the two screens below.





   Comment on the size of the range of volumes that are produced from this range of possible radii.
3. Jillian was measuring only the outside of the ball, yet the volume she wanted was *inside* the ball – the *capacity* of the ball. Suppose the leather was about 3 mm thick. Determine a range of values for the volume of air in the ball. Compare your answers with those of others and discuss any differences.
4. Use Jillian's method to find the amount of air inside a tennis ball and a table tennis ball. Compare your answers with those of other people.

Errors of measurement are always involved in practical situations, and a calculator is a useful tool for analysing them. Tim and Hshen were trying to find the height of a large tree near their school. They knew that they needed to measure the angle of elevation θ of the tree and their distance x from the tree.



Then the tree height can be calculated as

$$\text{height} = x \tan \theta .$$

Tim said that the length of his pace was fairly close to a metre, especially if he was careful to make each pace about the same. He decided to measure Hshen's distance to the tree by pacing.

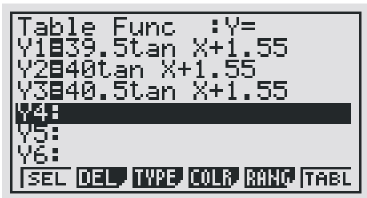
Hshen used her school protractor to measure the angle of elevation of the tree. She looked carefully along the straight edge and used a piece of string with a weight on it to make a vertical line. Hshen knew she was about 160 cm tall.

Interaction L

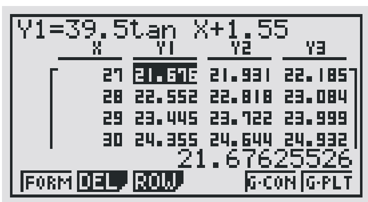
1. What likely errors of measurement are there in this situation? List as many as you can and compare your list with your partner's.
2. Notice in the diagram that the angle of elevation is measured from eye level, not from ground level. How should this be taken into account in doing the final calculations?

Tim thought that the distance to the tree was 40 metres, to the nearest metre. Hshen found it difficult to measure the angle of elevation accurately with her simple equipment. She decided that the angle was about 32°, but thought that the actual measurement could be 5° either side of 32°.

To look at the effects of these errors of measurement on their calculations, they decided to consider a range of values for both the distance and the angles, as shown below. On the calculator, X (rather than θ) has been used to refer to the angle of elevation. Notice also that Hshen's eye height has been added.

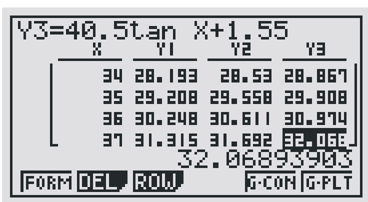


The tables show the possible values for the height of the tree.



The screens show a column for three possible values of the distance (39.5 m, 40 m and 40.5 m)

Each row of the tables shows a possible angle size, from 27° to 37°.



Tim and Hshen were both surprised that there was a very wide range of tree heights, depending on the data used: from 21.7 m up to 32.1 m.

Interaction M

1. Which error has more serious consequences for measuring the height of the tree: the distance or the angle? Justify your answer.
2. What result(s) do you think Tim and Hshen should report to describe the tree height?
3. Hshen and Tim also used the same method to find the height of a large cliff near their school. Tim paced out a distance of 30 m (±1 m) and Hshen measured an angle of 75° (±5°). What height would you give for the cliff? Will their results be more or less accurate than for the tree measurement? Explain your answer.
4. Take the necessary measurements to calculate the height of a tall object near your school, such as a large tree, an office building or a TV tower. Analyse carefully your errors of measurement and their effects.

Making decisions about accuracy

At the start of this book, we saw how the position of Chloe's pump changed the length of her pipeline and hence how the best position for the pump could be found. To do this, a mathematical model of the situation was made and we used mathematical analysis (relying on the Theorem of Pythagoras) to help find the shortest length of pipeline needed.

The calculator allows you to get a fairly accurate decimal approximation to questions of the kind in which Chloe was interested. Mathematics allows you to find exact answers to such questions – those which can be expressed using only whole numbers in fractions and radicals (or surds).

In this particular case, the exact solution to the problem involved placing the pump $1\frac{2}{3}$ of a kilometre from A, in which case the exact length of the pipeline turns out to be $\sqrt{34}$ km.

In practical situations like Chloe's, you have to make *decisions* about an appropriate level of accuracy. How close is close enough? Let's look again at Chloe's pump situation, thinking about the practicalities a little more carefully.

It is possible that exactly $1\frac{2}{3}$ of a kilometre from A may be an inconvenient place to put a pump. Perhaps the soil is a bit swampy there, or there is a tree in the way, or the pump would be too inaccessible for maintenance purposes. How much difference would it make to the price of the operation to have the pump a small distance away from the optimal position?

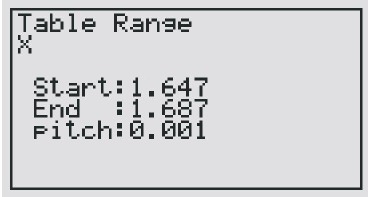
To the nearest metre, you saw previously that the shortest pipeline resulted from placing the pump 1.667 km (1667 m) from A. The pipeline was $\sqrt{34} \approx 5.831$ km (5831 m) long.

If we have to settle for positioning the pump within 20 metres of the optimal position, what would the effect be on the length of the pipe required and hence the cost of the exercise? To explore the possibilities look at the table of values again.

The screen here shows the same function as before for the length of the pipeline, entered in TABLE mode:

$$y = \sqrt{1 + x^2} + \sqrt{4 + (5 - x^2)}$$

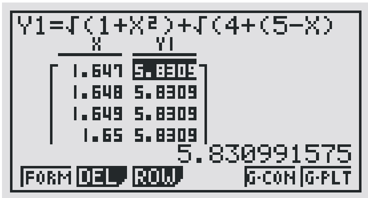
We'll start by finding the length of the pipeline if the pump is placed within 20 m of the best location.

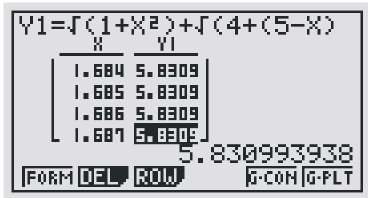


To do this, use the RANGE to tabulate values from 1.647 to 1.687, in steps of 0.001. (Remember that a metre is 0.001 km, so these values are within 20 m of the optimal position.)

The screen shows the settings.

Perhaps surprisingly, the resulting table shows that the pipeline length doesn't change (to the nearest metre) over this entire range of values for x:





Interaction N

1. Change the RANGE settings to explore the pipeline lengths within 50 m of the optimal pump location. What range of pipeline lengths result?
2. Explore further to find how far you can move the pump from its optimal location without increasing the pipeline length more than one metre.

In this case, Chloe is mainly concerned about the *costs* of constructing the pipeline. It is quite possible that the costs of locating the pump in the optimal place are greater than the costs of choosing a slightly less favourable location.

For example, if it cost \$150 to remove a tree so that the pump could be located there, in order to save one metre of pipeline (costing only \$15), she would be better advised to have a slightly longer pipeline.

Suppose that Chloe wants to know where the pump may be placed so that the cost of the pipe is within \$150 of the minimum possible cost. (That is, the total length of the pipe is at most 10 metres more than the minimum that is required.) The calculator can be used to determine the positions where the pump can be placed.

Interaction O

1. What will be the mimimum possible cost of the pipeline (when the pump is placed in the best position)?
2. Do you think a margin of error of \$150 is appropriate in this case? Explain why.
3. Use the TABLE mode to explore the situation further. What range of pump locations lead to a pipeline that is within \$150 of the optimal cost?
4. Suppose that Chloe was happy with a solution to her problem that was within 1% of the best pipeline price. Where can she put the pump?

Investigation:

Consider again the rugby kicking situation, which also involves some mathematical modelling. Decide upon the appropriate levels of accuracy for this situation.

Investigate the difference in kicking angle that results from changing the distance at which the kick is taken from the goal line. How accurately are the lengths measured in practice by rugby players (who don't usually have measuring tapes available to them during a game)?

ANSWERS

Some of the questions that have been asked do not have a single correct answer. In such cases, MPA (which stands for many possible answers) will be the answer supplied. In many cases some supporting comment is supplied.

Interaction A

1. $\sqrt{17} \approx 4.123$ km
2. $\sqrt{5} \approx 2.236$ km
3. approximately 6.359 km
4. MPA, eg,

Distance AP	Distance PB	Length of pipe from block 1 to pump (km)	Length of pipe from block 2 to pump (km)	Total length of pipe required (km)
2	3	$\sqrt{5} \approx 2.236$	$\sqrt{13} \approx 3.606$	5.842
1	4	$\sqrt{2} \approx 1.414$	$\sqrt{20} \approx 4.472$	5.886

5. MPA, some places are better than others. A position somewhere between $x = 1$ and $x = 2$ seems to be the best.
6. $5 - x$ km

Interaction B

1. 1.6667 km, 5.830951895 km
2. MPA, eg, such a high level of accuracy isn't needed here, or in other practical situations of this kind.
3. She could have saved about 528 m, with value $7920.

Interaction C

1. No, there are an infinite number of combinations.
2. Volumes are 100.0002, 99.9961 and 100.0044 cc respectively.
4. You won't be able to draw a diagram to the accuracy given, as pencil lines are too thick.
5. Height decreases as radius gets larger.
6. A 'middle' value of the radius seems most *sensible*, so that the cylinder is neither too 'thin' nor too 'fat'.

Interaction D

1. As $\dfrac{100}{\pi \times 0}$ is undefined.
2. Volume of hemisphere exceeds 100 cc, so cylinder must have negative volume to give combined volume of 100 cc. This makes no practical sense, as there will be no cylinder.
3. Seems difficult to do as no criteria for optima has been defined nor seem obvious.

Interaction E

1. 146.96 sq cm, 95.39 sq cm, 83.04 sq cm respectively.
3. General observation is that the surface area seems to decrease as the radius increases.

Interaction F

1.
$$2\pi r\left(\frac{100}{\pi r^2} - \frac{2}{3}r + r\right) = 2\pi r\left(\frac{300}{3\pi r^2} - \frac{2\pi r^3}{3\pi r^2} + \frac{3\pi r^3}{3\pi r^2}\right)$$
$$= 2\pi r\left(\frac{300 + \pi r^3}{3\pi r^2}\right)$$
$$= \frac{2}{3r}\left(300 + \pi r^3\right)$$

2. MPA. Maximum of 3.8 cm suggested as a hemisphere with this radius already has a volume greater then 100 cc (recall Interaction D). A radius of 3.6 cm is best.
3. 3.63 cm
4. –0.004 cm. This value would be zero if the exact value of the radius was used. Hence the structure would only be a hemisphere.
5. A hemisphere gives the minimum surface area of any solid for a given volume. Hence in this situation the cylinder does not exist ($h = 0$ cm) and the piston would have nowhere to reside.

Interaction G

1. AOC = 72.12°, BOC = 63.43°, AOB = 8.69°.
2. AOC = 57.17°, BOC = 45°, AOB = 12.17°.
3. AOC = 45.94°, BOC = 33.69°, AOB = 12.25°.
4. AOC = 37.78°, BOC = 26.57°, AOB = 11.21°.
5. AOC = 31.80°, BOC = 21.80°, AOB = 10.00°.
6. General observation is that as the distance from the goal line increases from zero, the room for error first increases and then decreases.

Interaction H

1. MPA, eg, professionals don't want to be too close to or too far way from the goals. Angle is about the same regardless of distance. It will differ for individuals.
2. 12.17°, 12.09°. This difference is too small to be of concern to a kicker.
3. 18 m
4. 23 m
5. 34 m

Interaction I

1. $9.2 \times 5.5 \div 2 = 25.3$ sq cm
2. Sides: 9.2 cm, 6.5 cm, 7.9 cm; heights: 5.5 cm, 6.4 cm, 7.7 cm
3. $6.5 \times 7.7 \div 2 = 25.0$ sq cm, $7.9 \times 6.4 \div 2 = 25.3$ sq cm
4. MPA. The difference is about 27 sq mm, the area of the square with each side about 0.5 cm.
5. Measurement errors include: people read their rulers differently; rulers are not (exactly) identical; rounding is not always the same; some measurement may have been a bit careless; etc. Measurements are made to nearest millimetre, so that results will differ.

Interaction J

1. In this case, $S = 15$ and the program *HERON* will evaluate

$$\sqrt{15\,(15-5)\,(15-12)\,(15-13)} = \sqrt{15 \times 10 \times 3 \times 2} = 30$$

2. 25.2 sq cm; the results are a little different because different methods are being used.
3. 24.8 sq cm
4. 25.5 sq cm

Interaction K

1. $V = \frac{4}{3}\pi \times 11^3 = \frac{5324\pi}{3}$
$\approx 5575.28\,\text{cc} \approx 5.6\,\text{litres}.$
2. Volumes vary from about 4189 cc to about 7238 cc, a very large range, although the original error of measurement did not seem very large.
3. Regard the radius as 10.7±1 cm, leading to volumes from 3823 cc to 6709 cc, which again is a very wide range of results.
4. MPA, depending on measurements and the actual balls used. For example, we found a tennis ball radius to be 3.0 ±0.3 cm and a table tennis ball to have radius 1.9±0.2 cm. In these cases, the volumes were 82-151 cc and 21-39 cc, each of which is a wide range. More careful measurements will reduce these ranges.

Interaction L

1. MPA, e.g. paces may be different sizes, ground may not be quite flat, may not be clear precisely where to start pacing from, hard to keep protractor still, tree may move a little in the wind, not always clear precisely where the tree top is, etc.
2. The height of Hshen's eyes from the ground should be added to the value calculated for height from the equation $\text{height} = x \tan \theta$.

Interaction M

1. The angle error is far more severe, since it is about 30% of the measurement. In the table, the range of values for each distance (column of the table) is quite large, while the range of values for each angle (row of the table) is fairly small (21.676-31.315 compared to 21.676-22.185).
2. Better to give an interval of heights, eg., 22-32 m or 27±5 m.
3. 81-178 metres. The results are less accurate. The distance is a little less accurately measured and the angle is measured to about the same level of accuracy, but the consequences of the errors are more dramatic.
4. MPA. Check your answers with those of other students.

Interaction N

1. Use Start and End values of 1.617 and 1.717 respectively. All the pipeline lengths are still 5831 m, measured to the nearest metre.
2. For x-values from 1.595 to 1.740, inclusive, the pipeline length is still within a metre of the optimal value. That is, the pump can be within 0.072 km or 72 m of the optimal position without increasing the pipeline length by more than a metre.

Interaction O

1. The lowest cost is $\$15 \times 1000 \times \sqrt{34} = \$87464.28$, to the nearest cent.
2. MPA, eg., many might think that it is a very small amount in relation to the total cost. It is only 0.17% of the total.
3. \$150 will pay for 10 m of pipeline. Find which values of x lead to y values within 0.01 km of the optimal value of 5.831 km. The range is from x = 1.37 to x = 1.98, to the nearest 0.01 km. That is, the pump can be about 300 m either side of the optimal location.
4. The range is now within \$875 or about 58 m of pipeline from the optimal. To the nearest 10 m, this is a range of x = 0.98 km to x = 2.48 km. This interval covers a distance of 1.5 km or 1500 m, within which the pump can be located – a very wide margin of error.