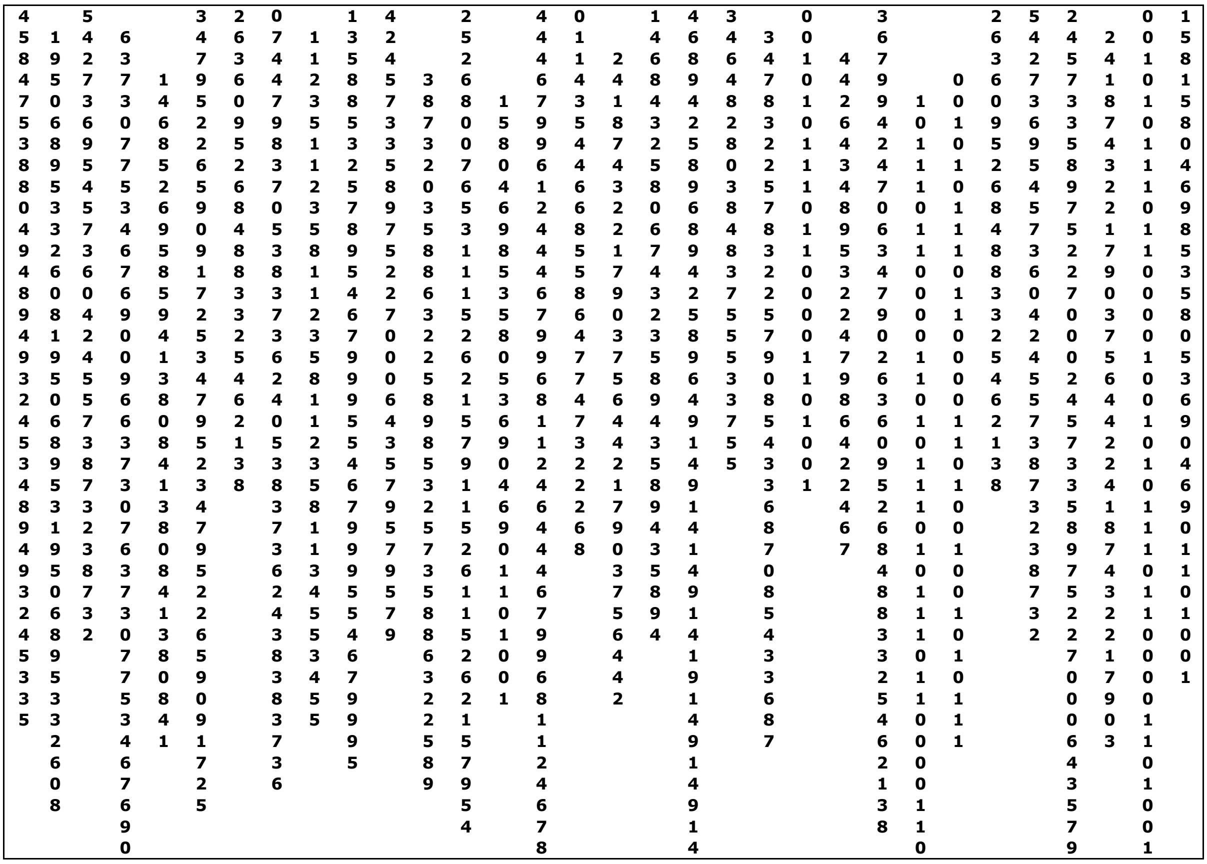
NUMBERS

ITS ORDER & STRUCTURE



By Martin Schmude & Anthony Harradine

Contents

INTRODUCTION

Numbers play a crucial role in society and as such, it is imperative to understand how numbers work. It seems the main reasons why it is important to be numerate are to build one's number sense, which can help you to enjoy life by being comfortable around numbers and have the skills to solve problems.

The journey through this unit will incorporate a large amount of problem solving, hoping to build competence not only in number sense, but problem solving ability as well.

Many students will have covered many, if not all, aspects of this course already, which begs the question why do we have to cover this again? This unit hopes to help students learn concepts more profoundly and consolidate these skills through challenges and interesting activities. It is also hoped to cater for all levels of ability.

The *CASIO CFX-9850GB PLUS* calculator will be used often, but only partly for calculations. It will be unsuitable to rely upon it for every answer which leads students to be extremely familiar with the four main operations, namely +, -, x, ÷ (a times table is located at the end of this unit if needed). The development of the programs called ADDs will be used significantly in many different ways – to hone skills and to aid problem solving. These particular skills will be enhanced throughout the unit by the use of MENTAL© program family.

Some of the activities from this booklet came from the National Council of Teachers of Mathematics. The applet program activities can be played on their excellent website. The address is

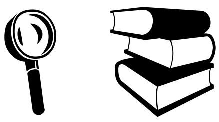
http://illuminations.nctm.org/imath/

Some of the anecdotes have come from the two books of Rob Eastaway and Jeremy Wyndham called *Why do Buses Come in Threes* and *How Long is a Piece of String*.

NUMBER FAMILIES

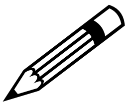
Integers are whole numbers, such as 1, 2, 3, ... These POSITIVE integers may be organised into different groupings or families depending on the rules we set. For instance, you all know about ODD and EVEN numbers. ODD and EVEN are two examples of types of families.

THEORY



An EVEN number is any number that can be divided in half evenly. ODD numbers cannot be divided evenly by two and occur in between even numbers.

QUESTIONS



1. For the integers 1 - 20, list the even and odd numbers.
2. What would be the simplest manipulation of any positive integer, *n*, so that it will always produce an even number? EXPLAIN your reasoning clearly.
3. What would be the simplest manipulation of any positive integer, *n*, so that it will always produce an odd number? EXPLAIN your reasoning clearly.

JUST FOR ☺ INTEREST

A man called Boethius actually went further to classify even numbers into three groups:

- *Evenly even*
- *Evenly odd*
- *Oddly even*

*An **evenly-even** number is one that can be divided into two equal parts, which can also be divided into two equal parts, which can also be divided into two equal parts, and so on until you reach 1. Example:*

32 = 16 + 16
16 = 8 + 8
8 = 4 + 4
4 = 2 + 2
2 = 1+1

*Likewise, an **evenly-odd** number is one that can be divided into equal parts, but the halves cannot be divided into equal parts. Example:*

22 = 11 + 11
11

***Oddly-even** numbers share properties of both evenly-even and evenly-odd numbers. An oddly-even number can be divided into equal parts, and divided again, but not to 1. Example:*

24 = 12 + 12
12 = 6 + 6
6 = 3 + 3
3

WHAT IS ARITHMETIC?

Throughout this unit, the four main operations in arithmetic will be used considerably. That is, *addition*, *subtraction*, *multiplication and division*. But are you entirely sure what they all mean?

Addition and *subtraction* are the simplest ways of combining things. It is a very easy concept to grasp because we use it every day, whether it be dealing with money or looking at the time. Because of the simplicity, no more needs to be said.

A way to think of *multiplication* is that of the repeated addition of a number. For example, the summation below could be rewritten in a different way.

$$\underbrace{3+3+3+3+3+3}_{\text{6 threes}} = 18 \qquad 6 \times 3 = 18$$

Notice also that the same summation could be thought in the opposite (converse) way.

$$\underbrace{6+6+6}_{\text{3 sixes}} = 18 \qquad 3 \times 6 = 18$$

Division then is simply the repeated subtraction of a certain number from another number. An example is given below.

$$40 - \underbrace{8-8-8-8-8}_{\text{5 eights}} = 0 \qquad 40 \div 8 = 5$$

Notice also that the same subtraction could be thought in the opposite (converse) way.

$$40 - \underbrace{5-5-5-5-5-5-5-5}_{\text{8 fives}} = 0 \qquad 40 \div 5 = 8$$

ASSIGNMENT

Pick a number between 10 and 100. Throughout this unit of work, you are to gather information about your number – any special properties it has, important historical value, why you chose it, connections it has to other numbers, etc. – and present at least a A4 page report at the conclusion of the unit.

MENTAL ARITHMETIC

CHALLENGE

How many 4 digit numbers can be made from the digits 2, 3, 5, 6 and 9.

What is the sum of all these numbers?

Although calculators are everywhere nowadays – even on our mobiles, it is still important to be competent with mental calculations. Why? Because it will improve your number sense, making you more comfortable with working with numbers.

Mental Multiplication

Probably the most helpful tip to remember when doing mental multiplication is the DISTRIBUTIVE LAW.

THEORY

DISTRIBUTIVE LAW says that:

$$(a+b)\times c = a\times c + b\times c$$

The simplest way to understand the distributive law is by using an example Imagine you have the following question.

$$73\times 6$$

We are looking to break this into two parts – both of which are easy to calculate. Any numbers that are multiples of 10 make for easy multiplication, so we can break 73 into 70 + 3. This leads to

$$\begin{aligned} 73\times 6 &= (70+3)\times 6 \\ &= 70\times 6 + 3\times 6 \end{aligned}$$

This looks much simpler and the answer is just

$$\begin{aligned} 73\times 6 &= (70+3)\times 6 \\ &= 70\times 6 + 3\times 6 \\ &= 420+18 \\ &= 438 \end{aligned}$$

QUESTIONS

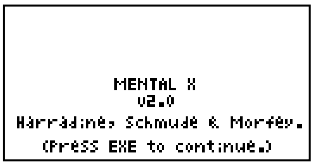
As mentioned before, working with multiples of 10 makes multiplication straightforward. Using the distributive law, or otherwise, break the following questions into simpler calculations and then compute your 'new' calculation.

1. 12×7
2. 26×11
3. 83×4
4. 49×6
5. 68×3
6. 88×3
7. 13×2
8. 52×12

CHALLENGE

Can you come up with rules for multiplying by 5, 9, 11 and 13?

Now have a try of the **MENTALX** ADD.



OUT OF CHAOS COMES A NEW ORDER

If you were presented with the following and asked to calculate the answer, what possible answers could you come up with?

$$24 \div 2 + 4 - 1 \times 3$$

No doubt you came up with quite a few. Assuming your number crunching is correct, none of your answers are incorrect, but as you could plainly see, this could lead to major problems with everybody coming up with different answers. Over the centuries, a certain order has been created, and it became especially important in the 1600's with the discovery of algebra and more recently with computer programming. We are going to use some logic to discover the *order of operations*.

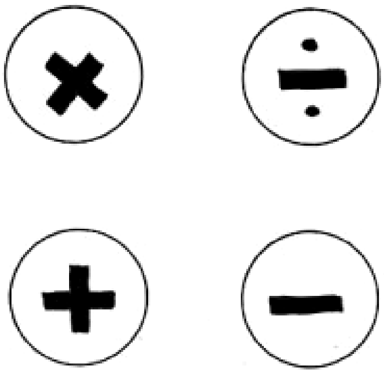
ACTIVITY

Your calculator uses the correct order of operations. Enter the following calculations and then determine which operation is completed first.

1. $30 \div 5 + 10$ ________ comes before ________
2. $20 + 5 \times 12$ ________ comes before ________
3. $12 - 3 \times 4$ ________ comes before ________
4. $50 - 48 \div 6$ ________ comes before ________
5. $10 \times 3 \div 6$ ________ comes before ________
6. $56 \div 8 \times 3$ ________ comes before ________
7. $15 - 6 + 2$ ________ comes before ________
8. $75 + 25 - 10$ ________ comes before ________

CONNECTION

If → means 'comes before' (for example, crawling → walking), copy the diagram below into your book and use it to clarify what you've discovered.



EXERCISES

Now try the exercise Order of Operations in your textbook.

THE PRODUCT GAME

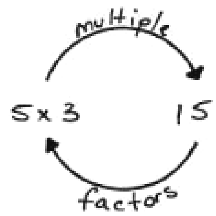
http://illuminations.nctm.org/imath/6-8/ProductGame/product1.html

THEORY

A number, *a*, is said to be a MULTIPLE of *b* if *b* divides *a* evenly.

Unlike the Factor Game where you start with the number and determine the factors, now we start with the factors and determine the product. This product is a multiple of both the factors. For example, these four sentences are all ways of expressing $5 \times 3 = 15$:

15 is a multiple of 5
15 is a multiple of 3
3 is a factor of 15
5 is a factor of 15



ACTIVITY

Play the Product Game several times with a partner. In the appendix at the back of this booklet on page 32, you will find the Product Game Board. Just like the Factor Game, take turns making the first move. Look for moves that give the best scores. In your book, note any strategies you find that help you to win.

RULES

1. Player A puts a marker on a number in the factor list. No space on the product grid is to be filled in with Player A's colour because only one factor has been marked; it takes two factors to make a product.

2. Player B puts another marker on any number in the factor list (including the same number marked by Player A). The space on the product grid containing the product of the two factors marked is coloured in with Player B's colour.

3. Player A moves *either one* of the markers to another number and the new product is filled in with Player A's colour.

4. Each player, in turn, moves a marker and colours the space of the product of the two selected numbers. If a product is already coloured, the player does not get a mark for that turn. The winner is the first player to mark four spaces in a row - up and down, across, or diagonally.

The Product Game Board

1	2	3	4	5	6
7	8	9	10	12	14
15	16	18	20	21	24
25	27	28	30	32	35
36	40	42	45	48	49
54	56	63	64	72	81

Factor List

1	2	3	4	5	6	7	8	9

QUESTIONS

1. On the Product Game board, suppose your filled-in squares are 16, 18, and 28, and your opponent's filled-in squares are 14, 21, and 30. The factor markers are on 5 and 6. It is your turn to move a marker.

The Product Game Board

1	2	3	4	5	6
7	8	9	10	12	14
15	16	18	20	21	24
25	27	28	30	32	35
36	40	42	45	48	49
54	56	63	64	72	81

Factor List

1	2	3	4	[5]	[6]	7	8	9

   (a) List the possible moves you could make.

   (b) Which move(s) would give you three squares in a row?

   (c) Which move(s) would allow you to block your opponent?

   (d) Which move would you make? Explain your strategy.

2. Using the words *factor*, *divisor*, *multiple*, *product* and *divisible by*, write as many sentences as you can about the equation $6 \times 3 = 18$.

3. Study the Product Game Board below.

9	15	18	
21	**?**	30	35
	36	42	49

   (a) What four factors were used to create the Game Board?

   (b) What is the missing number in the Board?

THE FACTOR GAME

CHALLENGE

How many zeros are on the end of the following numbers?

$12 \times 13 \times 14 \times 15 \times \ldots\ldots \times 19$

$12 \times 13 \times 14 \times 15 \times \ldots\ldots\ldots \times 20$

$12 \times 13 \times \ldots\ldots\ldots \times 99$

http://illuminations.nctm.org/imath/6-8/FactorGame/factor1.html

On a Free Sunday, four boys went into the City. Brett has $30 to spend, Matt has $20, Marty has $15 and Mark has $10. The following could be said about the amount of money Brett has with respect to the other boys.

Brett has:

$1\frac{1}{2}$ times more than Matt 2 times more than Marty 3 times more than Mark

$30 = 1\frac{1}{2} \times 20$ $30 = 2 \times 15$ $30 = 3 \times 10$

Notice that multiplying 2 whole numbers can create 30. We call these numbers *whole number factors* or *whole number divisors*. Although 20 is a whole number, it is not a whole number factor since there is not another whole number that it can be multiplied with to get 30. From now on, we will call whole number factors just *factors*.

ACTIVITY

Play the Factor Game several times with a partner. In the appendix at the back of this booklet on page 31, you will find the Factor Game Board. Take turns making the first move. Look for moves that give the best scores. In your book, note any strategies you find that help you to win. If necessary, you can use the ADD **FACTFIND**.

RULES

1. Player A chooses a number on the game board by colouring it.

2. Using a different colour, Player B colours all the proper factors of that number, except the number itself. For example, if Player A chooses 12 then Player B would colour in the numbers 1, 2, 3, 4 and 6. Even though 12 is a factor, it is not a proper factor.

3. Player B colours a new number and Player A colours in all the proper factors that are not already coloured.

4. The players take turns choosing numbers and colouring factors.

The Factor Game Board				
1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25
26	27	28	29	30

5. If a player chooses a number that has all the factors already coloured in, that player loses a turn and does not get the points for the number coloured.
6. The game ends when there are no numbers remaining with uncoloured factors.

7. Each player adds the numbers that they coloured in and the player with the greater total wins.

QUESTIONS

1. Is it better to go first or second? Why?

2. What is the best first move? What is the worst first move? Why?

3. How do you know when the game is over?

4. Which number will always eliminated in the first move? Write a sentence about this number with respect to the other numbers.

5. Long ago, people observed the sun rising and setting over and over at about equal intervals. They decided to use the amount of time between two sunrises as the length of day. They divided the day into 24 hours. Use what you know about factors to answer these questions:

   (a) Why is it more convenient choice than 23 or 25?

   (b) If you were to select a number different from 24 to represent the hours in a day, what number would you choose? Why?

PLAYING TO WIN

No doubt some of you discovered that there were better numbers to choose for your first move than other numbers. Did you discover which was the best first move? What move should you take after the first move? There is a branch of mathematics that deals with finding a strategy to maximise your chance of winning (without cheating) and it is called *game theory*.

ACTIVITY Make a table of all the possible first moves. Then write down all the proper factors, what score you would receive and what score your opponent would receive. Here is how the table might start.

First move	Proper Factors	My Score	Opponent's Score
1	1	Lose a turn	0
2	1	2	1
3	1	3	1
4	1, 2	4	3

QUESTION

Review the questions above and see if you still agree with your original answer.

SIEVE OF ERATOSTHENES

Eratosthenes was a Greek mathematician from Alexandria (Egypt) who lived from 276 to 194 BC. He was a historian, astronomer, poet and geographer. He is known as the first man to measure the circumference of the earth and in the arena of mathematics, he is known for his Sieve. The way it works is to take a set of numbers and then systematically eliminate all the multiples of the numbers, and by reason, we should only be left with the numbers that are not multiples of any other number (except 1). These numbers are special numbers called *primes*. They are used significantly in cryptography, which is the study of ciphers and are the foundations of every number (which we will discover later).

THEORY

A number is PRIME if it has no other proper factors except 1 and itself, ie. only two factors.

A COMPOSITE number has more than two factors.

ACTIVITY

In this activity, you must follow the following steps.

1) Cross out the number 1. That is not considered prime. Can you find out why?
2) Circle the next number 2, then eliminate all the multiples of 2.
3) Circle the next number 3, then eliminate all the multiples of 3.
4) Move on to the next 'untouched' number and eliminate its multiples.

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36
37	38	39	40	41	42
43	44	45	46	47	48
49	50	51	52	53	54
55	56	57	58	59	60
61	62	63	64	65	66
67	68	69	70	71	72
73	74	75	76	77	78
79	80	81	82	83	84
85	86	87	88	89	90
91	92	93	94	95	96
97	98	99	100	101	102

QUESTIONS

1. Why does this method only leave only prime numbers?
2. Did you notice any patterns when you were eliminating the numbers? What were they?
3. How do you know when you can go no further?

THE GAUNTLET

4. If there were *n* numbers in the Sieve, what number would I have to go to ensure that only primes were left?

PRIME FACTORS

CHALLENGE

How many factors does 16 have? What is the sum of all of them?

How many factors does 2592 have? What is the sum of all of them?

As you noticed in the two games played earlier. Most numbers can be broken down into the product of smaller numbers. For example, 42 can be expressed as 6×7. The furthest that every integer can be factored is to its unique *product of prime factors*.

The following is an example of the steps to express a number in terms of its prime factors.

THEORY

Step 1: Find the lowest prime that will divide 300 (in this case, 2).

$$\begin{array}{r|l} 2 & 300 \\ \hline & 150 \end{array}$$

Step 2: After dividing 300 by 2 to get 150, continue to divide the result by 2 until 2 is no longer a factor.

$$\begin{array}{r|l} 2 & 300 \\ \hline 2 & 150 \\ \hline & 75 \end{array}$$

Step 3: Move onto the next highest prime that divides the result (in this case, 3).

Step 4: So far so good! Now 3 does not go into 25, so then 5 is chosen (since it is the next prime) and the algorithm finishes when a prime is the last number left.

$$\begin{array}{r|l} 2 & 300 \\ \hline 2 & 150 \\ \hline 3 & 75 \\ \hline 5 & 25 \\ \hline & 5 \end{array}$$

Step 5: So 300 has been reduced to its prime factors and so can be written as a products of the primes, hence

$$\therefore 300 = 2 \times 2 \times 3 \times 5 \times 5$$
$$= 2^2 \times 3 \times 5^2$$

CONNECTION

If you have never seen this notation before, here is the explanation.

There is notation to describe when two of the same numbers are multiplied to each other. For example, if the calculation is 7×7 then it is written 7^2 and called '7 squared'. It tells us that there are two 7's multiplying together.

How do you think we would write $9 \times 9 \times 9 \times 9$?

BEWARE!

Do **NOT** misinterpret the meaning of a^2. Now 5^2 does **NOT** mean 5×2! We already have notation for that and it is 5×2! 5^2 means 5×5, that is, times the number by itself.

EXERCISES

1. Reduce the following to their prime factors
   (a) 36 (b) 50 (c) 88 (d) 120

2. Time to try the ADD **PFACTGAME**.

JUST FOR ☺ INTEREST

Modern Day Cryptography

Cryptography is a well-established science and we know that it has been in existence for 2000 years, thanks to many Greek historians. In the past 30 years, however, it has gone leaps and bounds, mainly due to the advent of the computer.

There have been many well-known examples of the successes and failures of cryptography, such as Mary Queen of Scots, Julius Caesar and Emperor Augustus, American Civil War, the German Enigma Machine in World War II and the Internet.

So what sort of ciphers are used today? Prior to the late 1970's, the only systems were ones where trust between the sender and receiver was needed.

Nowadays, the types of encryptions are called Public Key Systems. This is where both the algorithm and encryption key are available to the public. The mathematics of these systems are extremely complex – partly to do with understanding but especially because of the size of the numbers. So what follows is just a brief description of how these things are encrypted work.

Before we start, take a second to do the following calculations:

1. 11×17
2. *Find two whole numbers that multiply to give 437.*

How long did the first question take compared to the second question? A calculator takes a fraction of a second to calculate question 1, but question 2 requires trial and error. The answer is 19 and 23, which is a unique answer because both 19 and 23 are special numbers – primes.

This is essentially how the RSA system works. It is called RSA after Ravest, Shamir and Adelman came up with this system in 1978. Basically it works by secretly choosing two massive primes, say 100 digits each. Multiplying these two numbers together gives an even bigger number, maybe 200 digits long, call it N. Next, you have to choose another prime number, 50 digits will do, call it E. You then need to equate numbers (two digits) to the letters, so A=01, B=02, C=03, … Now imagine you want to encrypt the message I NEED HELP NOW. This converts the message into the number 0914050504080512161415 23, which is called the plaintext, call it P. Now this is where things start to get complex! The ciphertext, C, is the result of finding the remainder when P^E is divided by N. A computer can do it in a fraction of a second.

This is such a secure method because even if the 'interceptor' knew it was encrypted using the RSA system and knew the letter N, it would still take lifetimes for a computer to try and find those two 100 digit primes in order to decipher the message.

Damn there are some smart people in the world!

WHEN NUMBERS HAVE NUMBERS IN COMMON

ACTIVITY Using a game board similar to the Product Game Board used earlier, you are to play The Common Factor Game several times with another person. In the appendix at the back of this booklet on page 33 and page 34, you will find the Common Factor Game Board and scorecard respectively. Just like the other games, take turns making the first move and look for moves that give the best scores. In your book, note any strategies you find that help you to win. You can use the ADD **GCD** to help.

RULES

1. Player A picks two numbers then calculates the highest factor that is common to both numbers. That will be Player A's score for that turn. Both of the chosen numbers are then crossed out and are not used for the remainder of the game.

2. Player B then has a turn, exactly like Player A, but with two less numbers to choose from.

3. Play continues until no more numbers are left. Since there are 36 numbers, there should be 18 moves – 9 to each player. The player with the greatest total at the end wins.

The Common Factor Game Board					
1	2	3	4	5	6
7	8	9	10	12	14
15	16	18	20	21	24
25	27	28	30	32	35
36	40	42	45	48	49
54	56	63	64	72	81

QUESTIONS

1. Which number would always give the lowest score, making it the worst to choose?

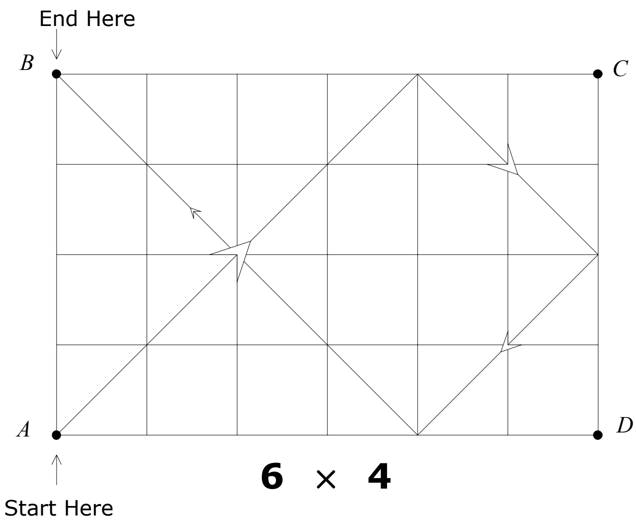
2. If the number 45 was the first number chosen, what other numbers could be chosen to give a score greater than 1?

3. From Q1, which move should be taken to achieve the highest score.

ACTIVITY It's time to play some Paper Pool. At the end of this unit, on page 35, you will find a whole set of pool tables of different dimensions. Below is a description of how to play the game.

RULES

1. The ball is to be hit from a corner pocket at a 45° angle.
2. Each contact with a wall produces a rebound of 45°.
3. The game continues until the ball lands in another corner pocket.
4. Count how many squares the ball travels before it is sunk.



6 × 4

CONNECTION

1. Can you make a conjecture about the link between the dimensions of the table and the distance the ball has to travel?

2. How squares would the ball travel through if the dimensions of the table were 8×6? 12×2? 10×5?

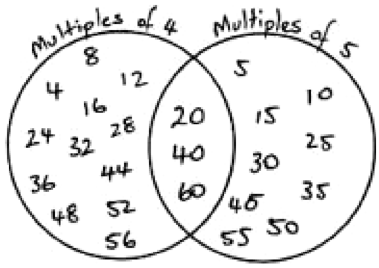
CHALLENGE

Explain why the tables have this property.

Paper Pool has a common theme with the following example. Mr Curran can run a 400m track in 4 mins. Mr Donohoe can run the same track in 5 mins. If they both run for an hour, when will they first meet? Perhaps the simplest way to solve this problem is to list the lap times (multiples).

For Mr Curran: 4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48, 52, 56, 60
For Mr Donohoe: 5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60

You may notice in the list above that there are multiples that occur in both Mr Curran's list and Mr Donohoe's list. These numbers can be called *common multiples* and we can display this information using a *Venn Diagram*.



A Venn diagram is an excellent way of displaying common and uncommon elements between sets.

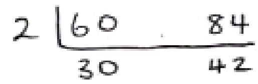
You can see from the Venn diagram that the multiples of 4 and 5 sometimes overlap, ie they have multiples in common. So to answer the question, Mr Curran and Mr Donohoe first meet after 20 minutes. This 20 is a special number in the relationship of 4 and 5 and it is called the *lowest common multiple* or *LCM*. As the name suggests, it is the lowest multiple which is common to both numbers.

LCM's are used significantly with fractions. Take the following situation. Imagine a pizza is cut into two pieces and another pizza is cut into three pieces. Now what is the smallest number of pieces that these pizzas can be cut into so that they have the same size pieces?

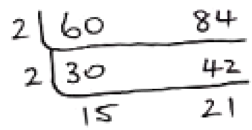
After thinking about for a second, you could cut the halves three times to make six pieces and cut the thirds in half to make six pieces also. Once again, this number is the LCM.

The algorithm used to find the LCM between two or more numbers is exactly the same for reducing a number to its prime factors. Once again, the algorithm will be explained through an example. So let's find the LCM of 60 and 84.

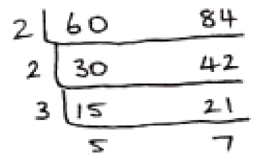
Step 1: Find the lowest prime that will divide 60 and 84 (in this case, 2).



Step 2: After dividing 60 and 84 by 2 to get 30 and 42 respectively, continue to divide the result by 2 until 2 is no longer a factor to either number.



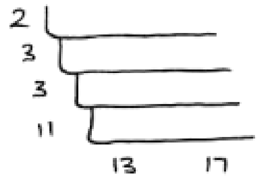
Step 3: Move onto the next highest prime that divides the results (in this case, 3).



Step 4: So far so good! Now there is no number that goes into 5 or 7 (except 1 of course), so then algorithm is finished.

QUESTIONS

1. Below is the shell of the algorithm. What were the two original numbers?



2. Using a Venn Diagram, write the multiples of 10 and 15 up to 80.

EXERCISES

3. Time to run some laps from your textbook.

TRIANGULAR NUMBERS

Numbers can make interesting patterns that often occur in real life, such as triangular numbers – flocks of birds and aircraft formations are a few examples. Ancient Greek mathematicians, particularly the Pythagoreans, first studied the properties of such numbers. We are going to investigate these numbers through the following problem.

Bowing for Business

The Japanese greet each other by bowing, particularly in business. There are many bowing techniques that range from a small nod of the head to a long, 90-degree bow. If the greeting takes place on a Tatami floor, people kneel down in order to bow.

The ritual is to bow deeper and longer than your opposite, if he or she is of higher social status than yourself. However, for foreigners it is usually sufficient to nod with the head, as most Japanese do not expect foreigners to know proper bowing rules, and a simple nod is usually preferable to an awkward bowing attempt.

Kazuo Kashio, the President of CASIO Japan, attends a meeting with the Board of Directors where there are 26 members.

QUESTIONS

1. Assuming each of the 27 individuals, bow to each other once, how many bows actually take place?

2. Complete the table

No. of people (n)	2	3	4	5	6
No. of bows (b)					

3. Consider the following pattern

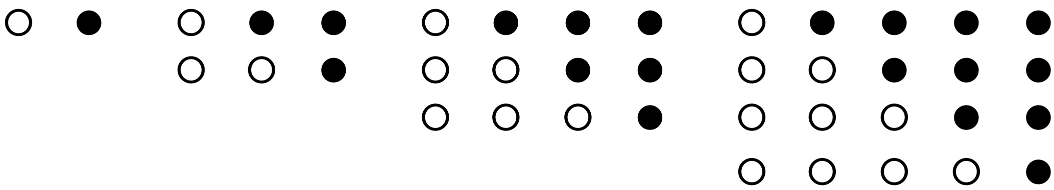
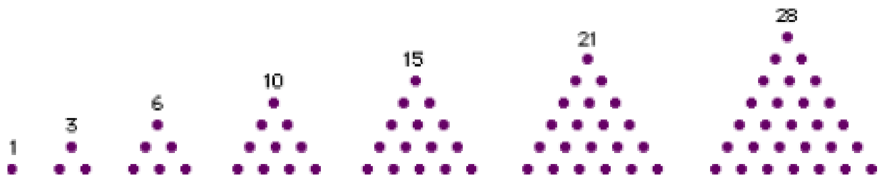


Figure 1 Figure 2 Figure 3 Figure 4

List the dimensions of each figure and then the number of circles in each figure (the area). The table below might help.

Length (l)	2	3	4	5	6
Width (w)					
No. of circles (n)					

4. Can you develop a general rule to describe the pattern of triangular numbers, using the pronumerals in question 3?



THE GAUNTLET

5. Can you discover a short cut way to find the sum of the integers from 1 to 100?

JUST FOR INTEREST

Packaging

The most analysed three-dimensional problem is that of stacking oranges. If you go to the fruit stall, you may well see oranges stacked in a pyramid fashion. If you visit a place with canons, you may well see canon balls stacked like the balls opposite.

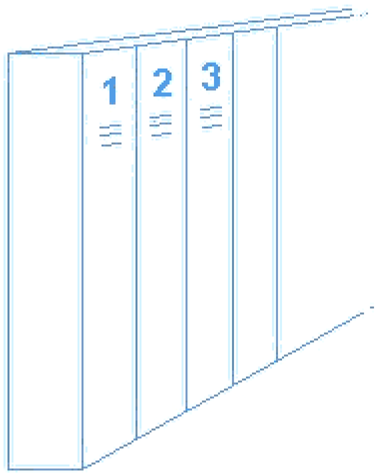
In 1690, the great astronomer Johann Kepler speculated that this was probably the most efficient way of stacking spherical objects (that is to achieve the least amount of air between spheres). Everybody knew it was the best way of stacking sphere because nobody was able to come up with a better way, but it wasn't until about 300 years later, in 1996, that it was finally proved true.

You may have noticed that the layers in this pyramid are the triangular numbers.

PERFECT SQUARES

Lockers

At St Joseph's College there are 1000 lockers — one per student. On the first day of the new school year the first student on the roll, Anthony Adams, runs through the school and opens every locker. The second student, Bart Baldwin, closes every second locker; the third student changes every third locker (opens it if it's closed, closes it if it's open); the fourth student changes every fourth locker — and so on until the 1000^{th} student either opens or closes the 1000th door.



ACTIVITY

1. How many doors are left open, and which ones are they?
2. What is the pattern to the numbers on the open doors?

THE GAUNTLET

3. Why are those particular doors the ones left open?

You may choose to use STAT mode on your graphics calculator. In LIST 1, enter 0 for an open locker and a 1 to represent a closed locker.

CHALLENGE

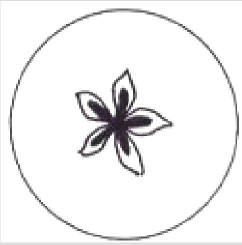
Prove that these particular numbers always have an odd number of factors.

JUST FOR ☺ INTEREST

How Nature Uses Numbers

Rob Eastaway and Jeremy Wyndham wrote about Fibonacci numbers in their book Why Do Buses Come in Threes. They started by asking the question why is it so hard to find a four-leaf clover? Well it seems that nature has chosen certain numbers to appear more than others. In the world of flowers, it seems one of the most common number of petals on a flower is five. Buttercups, mallow, pansies, primroses, rhododendrons, tomato blossoms, geraniums … are a few examples of flowers that have picked the number five.

Seeds also seem to come in fives. To see, just cut an apple in half (along the equator, which is in between the two ends. It will look like this:



Other things use Fibonacci numbers, like pears and pineapples. The number of spirals on a cone and pineapple usually have 8 one way and 13 the other, or 13 and 21, or 21 and 34, or 34 and 55, or 55 and 89.

The Italian Leonardo Fibonacci (1170-1240) didn't come across these special numbers by looking at flowers and fruit. He discovered the series when he was working out how many rabbits he would have if they bred at a particular rate.

So why is it that Fibonacci numbers crop up so often in plants?

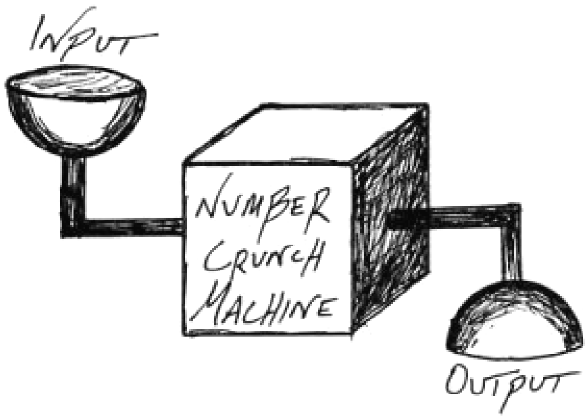
It all comes down to a link between the Fibonacci series and a special number that ancient civilisations believed to have divine and mystical powers. That special number is the golden ratio. You literally have the golden ratio within you! You might want to find out what this means.

NUMBERS IN NATURE

When a whole number 1, 2, 3, … is fed into the number crunch machine, two things happen:

(a) If the input number is even, the machine divides the number by two.

(b) If the input number is odd, the machine subtracts one from the number.



QUESTIONS

1. If the input was 17, what was the output?

2. If the input was 32, what was the output?

3. If the output was 6, what can be said about the input?

Suppose we feed the output back into the input. We continue to do so until the output reaches zero.

Thus, starting with an input of 5 we would get 5 ⇒ 4 ⇒ 2 ⇒ 1 ⇒ 0. Notice that **four steps** (shown as ⇒) are needed to reach zero.

QUESTIONS

1. How many steps would each of these numbers take?
   (a) 12
   (b) 32

2. Suppose we call 5 a **four-step** number
   (a) How many one-step numbers are there?
   (b) How many two-step numbers are there?
   (c) How many five-step numbers are there?

   You can use the **CRUNCHY** ADD if you wish. Can you think of a different way to find the solution?

No. of Steps	1	2	3	4	5	6	7	8
No. of numbers								

3.
   (a) Complete the table
   (b) Describe the pattern for the number of numbers.

BACKWARDS IS FORWARDS?

CHALLENGE

What is the 1000th palindromic number?

There are special words in the English language, like a place name in Adelaide called Glenelg and the title for a woman - Madam. Can you see what is so special about them? Can you think of any others?

Just as there are special words in the English language, there are special numbers in Mathematics, such as 1265648394047374049384656 21! These words and numbers are called *palindromes*. They are words and numbers that are the same forwards as they are backwards.

ACTIVITY

There is a simple way to create palindromic numbers. Start with any non-palindromic number you wish. Simply add the reverse of this number. Is the new number a palindrome? If not, do the same thing with the new number. Repeat this until the number is a palindrome. An example is shown below.

```
157+751
                 908
Ans+809
                1717
Ans+7171
                8888
```

QUESTIONS

1. Write down 5 3-digit palindromes.
2. Write down 5 5-digit palindromes.
3. If you have a 2-digit palindrome and assigned a letter to the same digits, the number could be written as bb.

   When you write that in expanded form, it is $10b + b = 11b$.

   What number would always divide a 2-digit palindrome?
4. Imagine you had a 4-digit palindrome, write it terms of letters. What number always divides a 4-digit palindrome?
5. Can you make a conjecture about a factor of even-digit palindromes?

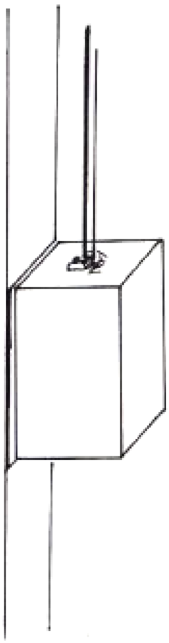
INTRODUCTION TO THE OPPOSITES

You don't have to be a genius to figure out the opposite of addition is subtraction and the opposite of multiplication is division (the reverse is also true). But what is the opposite of squaring a number?

ACTIVITY

One of the most fun things to do when you are young is to take a ride in a glass elevator. To see the ground speed away as the you feel yourself get heavier due to the acceleration. Then to have a sensation of weightlessness as the capsule slows down to a stop.

But what is fun for youngsters is a very serious business to adults. Elevators are the veins of buildings and they are required to run extremely efficiently. So how efficient are they expected to be? Well according to the company Otis, most people start to become frustrated after 15 seconds and even the most patient people start to consider the service poor after 35 seconds!



What's the solution? A very popular response is simply to makes the things go faster, but unfortunately the speed of the elevator is constrained by one thing … how people feel.

The maximum speed that people can tolerate is about 10 metres per second (ms^{-1}). That means it covers 10 metres every second. Since it starts from a stationary point, it has to accelerate to this top speed and people feel they are on a roller coaster if it exceeds 1 metre per second per second (ms^{-2}). This means the capsule gets faster by 1 metre per second every second.

Now if the elevator accelerates at 1 ms^{-2} to get to 10 ms^{-1}, it will take 10 seconds.

There is a rule that is used to figure out what distance it takes to accelerate to the top speed. It is:

$$\text{Distance travelled} = \tfrac{1}{2} \times \text{acceleration} \times (\text{time taken})^2$$

So the distance would be $\frac{1}{2} \times 1 \times 10^2 = 50$ metres, which is about 20 floors and it is assuming it has a clear run! Also, it needs another 20 floors to slow down, which means it would only be at top speed for a fraction of a second. Seems making the capsule faster doesn't help too much.

JUST FOR INTEREST

Approach from a different angle

There was someone who approached the elevator problem with some lateral thinking.

She thought one of the best ways to improve the elevator service was to put mirrors around the waiting areas. This led to people using the time to groom themselves by fixing their make-up or combing their hair. Hey, if it makes people happier!

The same rule that we used for the elevators can be applied to projectiles.

Niagara Falls is one of the most spectacular sights in the world. It separates the United States and Canada and during its history, there are 12 reported cases of people going over the falls. Most people usually did it in a barrel or some other type of protective gear.

In 1960, a boy, a girl and their uncle were fishing upstream from the falls when suddenly the boat capsized. They began to float downstream towards the falls at an alarming rate, while they desperately tried to swim to the shore. Both the uncle and girl reached it to the shore but unfortunately the boy didn't make in time and fell the 55-metre drop over the falls. His only protection was a life jacket and somehow, he miraculously survived the drop.

What we are going to calculate is how long was the boy in the air during his fall?

Using the rule that we had before, the distance travelled would be 55 metres, and the acceleration would be 10 ms^{-2} (this is a close approximation used to make the calculations simpler). Substituting this into our *formula* gives

$$55 = \tfrac{1}{2} \times 10 \times (\text{time taken})^2$$

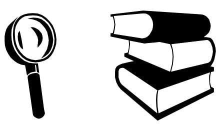
Unlike the elevator problem, this time we are finding the time taken. Now for the above *equation* (a mathematical sentence with an equal sign) to make sense, the $(\text{time taken})^2$ part must equal 11. If you abbreviate the (time taken) part into the letter *t*, we could then write:

$$t^2 = 11$$

So to solve the problem, a number has to be found that, when multiplied by itself, has to equal 11. Have a guess at what you think the answer is, even if you know the exact way to solve it.

Essentially what you have found is the opposite of squaring a number, called the square root. To explain this using an example, $8 \times 8 = 8^2 = 64$, so the square root of 64 is 8. The sign used for the square root is $\sqrt{\ }$, and so it can be written that $\sqrt{64} = 8$. They can also be called *surds*.

THEORY



If $a^2 = b$, then $a = \sqrt{b}$.

Just to make sure you are clear on what square roots are, have a go at the following exercises.

EXERCISES

1. Complete the following

$1^2 = 1$ $\sqrt{1} =$

$2^2 = 4$ $\sqrt{4} =$

$3^2 =$ $\sqrt{9} =$

$\vdots$ $\vdots$

$11^2 =$ $\sqrt{121} =$

$12^2 =$ $\sqrt{144} =$

2. Between which two consecutive integers do the following square roots lie?

(a) $\sqrt{6}$ (b) $\sqrt{45}$ (c) $\sqrt{73}$

(d) $\sqrt{109}$ (e) $\sqrt{27}$ (f) $\sqrt{52}$

3. Play the program SURDS and what the lowest score you can get.

The following questions are building skills that will be required to complete the Niagara Falls question.

4. What is the simplest way to find the middle number between 12 and 23? Can you write a general rule to find the midpoint of any two numbers?

5. Using the rule that you discovered in Q4, find the midpoint between:

(a) 25 and 48 (b) 43.2 and 43.7 (c) 2.03 and 2.57

6. Calculate the following

(a) $7^2 = 49$, $49 \div 7 = \ldots\ldots$ (b) $2.3^2 = 5.29$, $5.29 \div 2.3 = \ldots\ldots$ (c) $4^2 = 16$, $16 \div 4 = \ldots\ldots$

Now I know the answers to Q6 are not earth shattering but it is there just to show you that when a squared number is divided by the number that produces it, the answer is the number itself.

Going back to the boy falling over Niagara, to complete the problem, we have to find the exact answer of $\sqrt{11}$. You had a guess probably to a couple of decimal places, but how do you find the exact value? We are going to find out the answer using a system, which follows precise steps, called an *algorithm*. Having completed the questions above, the mathematics should make sense. To follow the algorithm below, read the observation then the next step taken.

Observation: $\sqrt{11}$ must lie between 3 and 4. Dividing 11 by **3** gives **3.6667** (4 decimal places)

Observation: **3** is not the square root of 11. This is because the answer is not 3 but **3.6667**.

Observation: The exact answer must lie between 3 and 3.6667 since $\left(3^2 = 9, \quad 3.6667^2 = 13.4447\,(4\text{ dp})\right)$. Find the midpoint of **3** and **3.6667**, which is **3.3334**. Divide 11 by **3.3334**, which gives **3.2999**!

Observation: This is getting much closer to the answer because **3.3334** is close to **3.2999**. So the exact answer must lie between **3.2999** and **3.3334**.

Observation: Find the midpoint of 3.2999 and 3.3334, which is **3.3167** (4 dp). Divide 11 by **3.3167**, which gives **3.3165**!

Observation: Continuing the steps above gives the answer **3.3166**. It seems the actual answer is **3.316**6 … to 4 decimal places. We have been rounding the off to 4 decimal places each time which will produce a certain number.

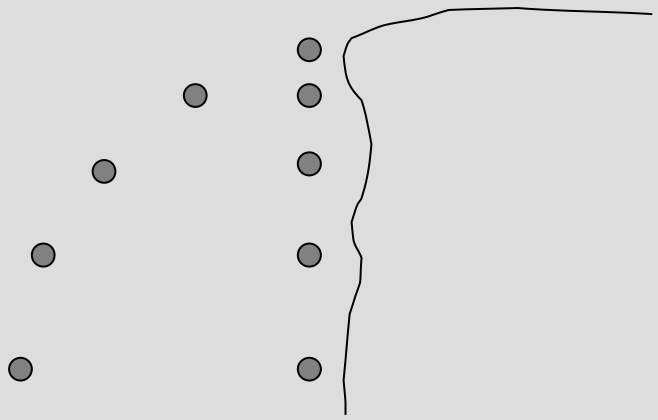
Observation	Action
$\sqrt{11}$ must lie between 3 and 4.	Dividing 11 by **3** gives **3.6667** (4 decimal places)
3 is not the square root of 11. This is because the answer is not 3 but **3.6667**.	
The exact answer must lie between 3 and 3.6667 since $\left(3^2 = 9, \quad 3.6667^2 = 13.4447\,(4\text{ dp})\right)$.	Find the midpoint of 3 and 3.6667, which is **3.3334**
	Divide 11 by **3.3334**, which gives **3.2999**!
This is getting much closer to the answer because **3.3334** is close to **3.2999**.	
The exact answer must lie between **3.2999** and **3.3334**.	Find the midpoint of 3.2999 and 3.3334, which is **3.3167** (4 dp)
	Divide 11 by **3.3167**, which gives **3.3165**!

JUST FOR ☺ INTEREST

Hang on, that's not right!

You can use the same formula to find the height of something. All you need is an object that is not too effected by air resistance – like a tennis ball. Simply drop it from the height and measure how long it takes. Gravity inflicts a vertical acceleration of 9.8 ms^{-2}*.*

You may have noticed that the two projectiles, namely the ball and the boy, are not moving in the same path. The ball has a vertical path (line) and the boy is moving along a curve. Surely the boy will take longer because he has to travel further. Well believe it or not, if you dropped a ball vertically from Niagara Falls just as the boy went over, they would land precisely at the same time (ignoring air resistance).



If you really don't think it's true, try it. Roll something off a ledge and just as it starts to fall, drop something from the same height.

BEWARE!

Be safe and make sure it is not dangerous to drop something.

APPENDIX

The Factor Game Board

1	2	3	4	5
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6	7	8	9	10
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11	12	13	14	15
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16	17	18	19	20
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21	22	23	24	25
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26	27	28	29	30
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The Product Game Board

1 □□□□□□□	2 □□□□□□□	3 □□□□□□□	4 □□□□□□□	5 □□□□□□□	6 □□□□□□□
7 □□□□□□□	8 □□□□□□□	9 □□□□□□□	10 □□□□□□□	12 □□□□□□□	14 □□□□□□□
15 □□□□□□□	16 □□□□□□□	18 □□□□□□□	20 □□□□□□□	21 □□□□□□□	24 □□□□□□□
25 □□□□□□□	27 □□□□□□□	28 □□□□□□□	30 □□□□□□□	32 □□□□□□□	35 □□□□□□□
36 □□□□□□□	40 □□□□□□□	42 □□□□□□□	45 □□□□□□□	48 □□□□□□□	49 □□□□□□□
54 □□□□□□□	56 □□□□□□□	63 □□□□□□□	64 □□□□□□□	72 □□□□□□□	81 □□□□□□□

Factor List

1	2	3	4	5	6	7	8	9

The Common Factor Game Board

1	2	3	4	5	6
□ □ □ □ □ □ □	□ □ □ □ □ □ □	□ □ □ □ □ □ □	□ □ □ □ □ □ □	□ □ □ □ □ □ □	□ □ □ □ □ □ □
7	8	9	10	12	14
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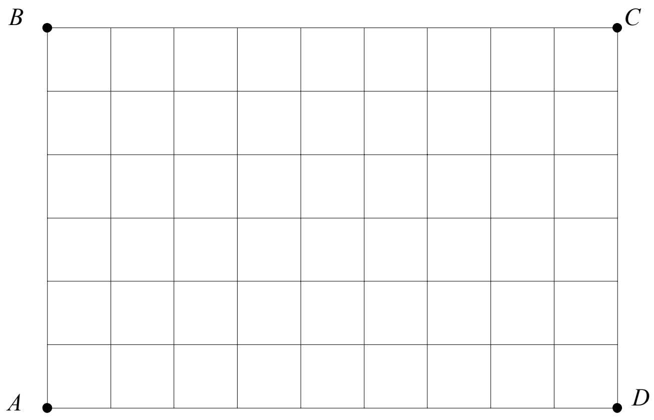
15	16	18	20	21	24
□ □ □ □ □ □ □	□ □ □ □ □ □ □	□ □ □ □ □ □ □	□ □ □ □ □ □ □	□ □ □ □ □ □ □	□ □ □ □ □ □ □
25	27	28	30	32	35
□ □ □ □ □ □ □	□ □ □ □ □ □ □	□ □ □ □ □ □ □	□ □ □ □ □ □ □	□ □ □ □ □ □ □	□ □ □ □ □ □ □
36	40	42	45	48	49
□ □ □ □ □ □ □	□ □ □ □ □ □ □	□ □ □ □ □ □ □	□ □ □ □ □ □ □	□ □ □ □ □ □ □	□ □ □ □ □ □ □
54	56	63	64	72	81
□ □ □ □ □ □ □	□ □ □ □ □ □ □	□ □ □ □ □ □ □	□ □ □ □ □ □ □	□ □ □ □ □ □ □	□ □ □ □ □ □ □

Common Factor Game Score Sheets

	Player A			Player B		
Pair	No. 1	No. 2	HCF	No.1	No.2	HCF
1						
2						
3						
4						
5						
6						
7						
8						
9						
		Total			Total	

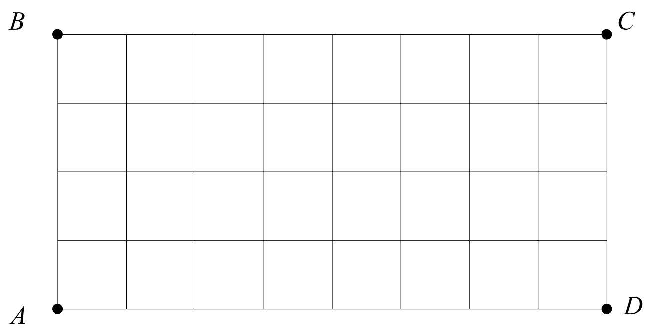
	Player A			Player B		
Pair	No. 1	No. 2	HCF	No.1	No.2	HCF
1						
2						
3						
4						
5						
6						
7						
8						
9						
		Total			Total	

Paper Pool

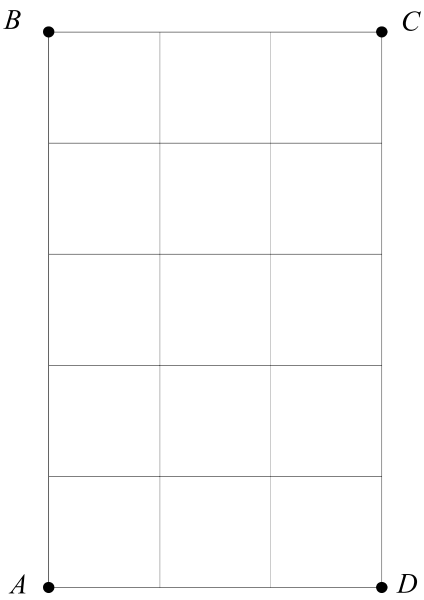


9 × 6

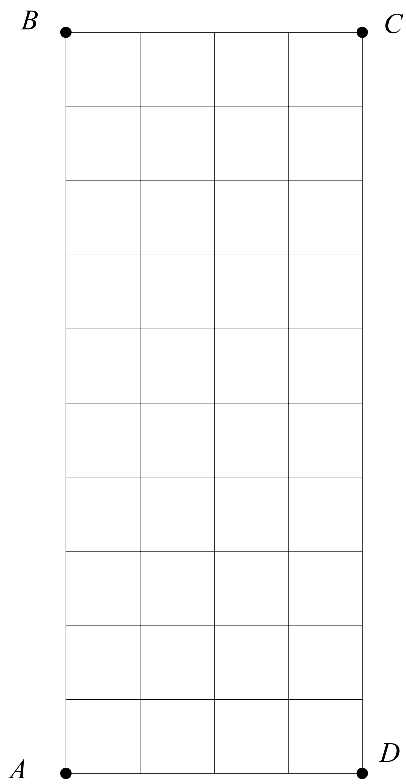
4 × 3



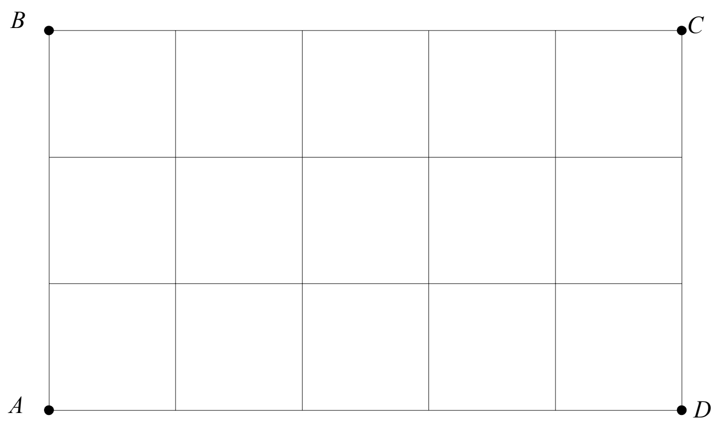
8 × 4



3 × 5



4 × 10



5 × 3

Times Tables

$1\times0=0$	$2\times0=0$	$3\times0=0$	$4\times0=0$
$1\times1=1$	$2\times1=2$	$3\times1=3$	$4\times1=4$
$1\times2=2$	$2\times2=4$	$3\times2=6$	$4\times2=8$
$1\times3=3$	$2\times3=6$	$3\times3=9$	$4\times3=12$
$1\times4=4$	$2\times4=8$	$3\times4=12$	$4\times4=16$
$1\times5=5$	$2\times5=10$	$3\times5=15$	$4\times5=20$
$1\times6=6$	$2\times6=12$	$3\times6=18$	$4\times6=24$
$1\times7=7$	$2\times7=14$	$3\times7=21$	$4\times7=28$
$1\times8=8$	$2\times8=16$	$3\times8=24$	$4\times8=32$
$1\times9=9$	$2\times9=18$	$3\times9=27$	$4\times9=36$
$1\times10=10$	$2\times10=20$	$3\times10=30$	$4\times10=40$
$1\times11=11$	$2\times11=22$	$3\times11=33$	$4\times11=44$
$1\times12=12$	$2\times12=24$	$3\times12=36$	$4\times12=48$

$5\times0=0$	$6\times0=0$	$7\times0=0$	$8\times0=0$
$5\times1=5$	$6\times1=6$	$7\times1=7$	$8\times1=8$
$5\times2=10$	$6\times2=12$	$7\times2=14$	$8\times2=16$
$5\times3=15$	$6\times3=18$	$7\times3=21$	$8\times3=24$
$5\times4=20$	$6\times4=24$	$7\times4=28$	$8\times4=32$
$5\times5=25$	$6\times5=30$	$7\times5=35$	$8\times5=40$
$5\times6=30$	$6\times6=36$	$7\times6=42$	$8\times6=48$
$5\times7=35$	$6\times7=42$	$7\times7=49$	$8\times7=56$
$5\times8=40$	$6\times8=48$	$7\times8=56$	$8\times8=64$
$5\times9=45$	$6\times9=54$	$7\times9=63$	$8\times9=72$
$5\times10=50$	$6\times10=60$	$7\times10=70$	$8\times10=80$
$5\times11=55$	$6\times11=66$	$7\times11=77$	$8\times11=88$
$5\times12=60$	$6\times12=72$	$7\times12=84$	$8\times12=96$

$9\times0=0$	$10\times0=0$	$11\times0=0$	$12\times0=0$
$9\times1=9$	$10\times1=10$	$11\times1=11$	$12\times1=12$
$9\times2=18$	$10\times2=20$	$11\times2=22$	$12\times2=24$
$9\times3=27$	$10\times3=30$	$11\times3=33$	$12\times3=36$
$9\times4=36$	$10\times4=40$	$11\times4=44$	$12\times4=48$
$9\times5=45$	$10\times5=50$	$11\times5=55$	$12\times5=60$
$9\times6=54$	$10\times6=60$	$11\times6=66$	$12\times6=72$
$9\times7=63$	$10\times7=70$	$11\times7=77$	$12\times7=84$
$9\times8=72$	$10\times8=80$	$11\times8=88$	$12\times8=96$
$9\times9=81$	$10\times9=90$	$11\times9=99$	$12\times9=108$
$9\times10=90$	$10\times10=100$	$11\times10=110$	$12\times10=120$
$9\times11=99$	$10\times11=110$	$11\times11=121$	$12\times11=132$
$9\times12=108$	$10\times12=120$	$11\times12=132$	$12\times12=144$

Tips to remember the table:

♣ Look for patterns ♦ Look, cover, write, check ♥ Practice, practice, practice