Optimisation
One ... Many

Greatest ... Least
Optimisation

Version 1.00 – June 2011

Written by Anthony Harradine and Alastair Lupton

Copyright © Harradine and Lupton 2011.

Copyright Information.
The materials within, in their present form, can be used free of charge for the purpose of facilitating the learning of children in such a way that no monetary profit is made.

The materials cannot be used or reproduced in any other publications or for use in any other way without the express permission of the authors.
1. **The cut straw**

Consider the common drinking straw (even better - get your hands on one). The one ‘illustrated’ here is 21 cm long.

Cut the straw (or imagine cutting the straw) some distance along its length.

a. i. If the cut is made 6 cm from the top, write down the calculation that determines the length of the ‘lower’ piece of straw. **Do not evaluate this calculation.**

   ii. If the cut is made 12 cm from the top, write down the calculation that determines the length of the ‘lower’ piece of straw. **Do not evaluate this calculation.**

Now make a ‘tee’ with the two pieces (as shown above right) and consider the triangle formed by the vertices of the ‘tee’.

b. i. Suppose the cut is made 6 cm from the top of the straw. Write down the calculation that would determine the area of this triangle. **Do not evaluate it.**

   ii. Suppose the cut is made 12 cm from the top of the straw. Write down the calculation that would determine the area of this triangle. **Do not evaluate it.**

   iii. Suppose the cut is made 18 cm from the top of the straw. Write down the calculation that would determine the area of this triangle. **Do not evaluate it.**

c. Describe what is the same and what is different about the three area calculations that you have written down in part (b).

d. Evaluate the area calculations that you have written down in part (b).

   As the distance from the top of the straw to the cut increases, what can you say about the area of the triangle?

e. Write down a calculation for the area of the triangle if the cut is made any distance from the top of the straw (i.e. $x$ cm).

f. Use this calculation to generate, with the aid of technology, a table of values for the area of the triangle for lots of $x$ values.

   Use your table of values to

   i. Check your answers to part (d).

   ii. Find the area of the triangle when $x = 7.5$.

   iii. Find the area of the triangle when $x = 14.1$.

   iv. Find the value of $x$ when the area of the triangle is 40 cm$^2$.

   v. Find the value of $x$ for which the area of the triangle is greatest.
2. **The pig pen**

Three adjoining, rectangular pig pens are to be built. 40 metres of fencing material is available. The area of the pens is important for pig health and well-being.

a. If side AD is made with 2.5 metres of fencing (and similarly for the sides parallel to it), write down calculations for:
   i. The amount of fencing material remaining to be used for sides AB and DC.
   ii. The length of AB.
   iii. The area of one of the pig pens.

b. Repeat part (a) if 5 metres of fencing is used for side AD.

c. Repeat part (a) if 7.5 metres of fencing is used for side AD.

d. Repeat part (a) if x metres of fencing is used for side AD.

e. Use your answer to part (d) to generate a table of values for the area of one of the pig pens when x takes lots of values.

f. What range of values can x take?

g. Use your table of values to find
   i. The value of the area calculations you wrote down in parts (a) to (c).
   ii. The length of AD if the area of one of the pig pens is 15.36 m².
   iii. The length of AD if the area of one of the pig pen is maximised.

h. Make a scale drawing of the pig pens when their area is maximised.

3. **The folded page**

Consider a piece of A4 paper with dimensions 21 cm by 29.7 cm. Looking at it in ‘landscape’ orientation, fold the top left corner down so that it touches somewhere along the bottom edge. Now consider the triangle ABD as shown in the diagram.

a. If AD is 5 cm, write down calculations for
   i. The length of DB.
   ii. The length of AB.
   iii. The area of triangle ABD.

b. Repeat part (a) if AD is 10 cm.

c. Repeat part (a) if AD is 15 cm.

d. Repeat part (a) if AD is x cm.
e. Use your answer to part (d) to generate a table of values for the area of triangle ABD for a range of values of $x$.

f. Use this table of values to find
   i. When the area of triangle ABD is greatest.
   ii. When the area of triangle ABD is least.

5. **The ‘swing-wing’ fighter plane**

Many modern fighter planes have ‘swing wings’. These wings swing out to increase lift during take-off and swing back to reduce drag during flight.

The simplest model of a pair swing wings looks like an isosceles triangle when viewed from above.

Consider a fighter plane that has swing wings with 10 metre leading edges as shown.

a. If AD is 3 m, write down a calculation for the area of triangle ACB.

b. If AD is $x$ m, write down a calculation for the area of triangle ACB.

c. Generate a table of values that calculates the area of triangle ACB for $x = 0, 1, 2, \ldots 10$.

d. Which of these $x$-values gives the largest value for the area of triangle ACB?

e. Refine your answer to part (d) to two decimal places.

f. What can you conclude about the largest possible area of triangle ACB?

5. **A graphical view**

In questions 1 to 4 you wrote down a function for an area in terms of $x$, a length.

For each of these functions:

a. Draw a clear graph of the functions, using an appropriate set of axes.

b. Locate the maximum value for the area, and the corresponding $x$ value.
6. **The two pipe problem**

Two homesteads need to draw water from a straight water pipe.

A pump is to be built somewhere along the water pipe so that it can pump water, via two straight pipes, to the two homesteads.

The homesteads are located such that the perpendicular distance of H1 from the water pipe is 1.6 km (i.e. AH1 is 1.6 km long) and that the perpendicular distance of H2 from the water pipe is 1 km (i.e. AH2 is 1 km long). The length of pipe AB is 3 km long.

a. If the pump was situated 1 km from the point A, write down a calculation for the total length of the two pipes that are needed.

b. If the pump was situated x km from the point A, write down a calculation for the total length of the two pipes that are needed.

c. Hence generate a table of values showing the total length of the two pipes needed for x = 0, 0.5, 1, 1.5, … 3.

d. Which of these x-values gives the shortest length of pipe needed?

e. Refine your answer to part (d) to the nearest metre.

f. To provide some installation flexibility, it is decided to plan for sufficient pipe so that up to 10 metres more than the shortest possible length was available to complete this job. Describe the range of possible locations for the pump.

7. **The athletics field**

An athletics field is being designed. Around its perimeter is to be a 400-metre running track, and its interior is to be grassed and used for field events. The ends of the field are semi-circular in shape, joined by two parallel straights.

a. If the semi-circular ends have a radius of 25 metres, write down a calculation for
   i. the length of the straight.
   ii. the area of the grassed interior of the athletics field.

b. If the semi-circular ends have a radius of r metres, write down a calculation for
   i. the length of the straight.
   ii. the area of the grassed interior of the athletics field.

c. Generate a table of values for A when r takes a range of values.

d. Hence or otherwise determine the dimensions of the athletics field when the grassed area is
   i. greatest.
   ii. least.
8. **The tin tray**

A rectangular sheet of tin measuring 90 cm by 60 cm is going to be made into an open topped tray.

To do this four equal-sized squares will be cut out of the corners of the sheet as shown.

The four “flaps” that this creates will be folded up, forming the tray.

a. What is the size of the biggest square that can be cut out? And the smallest?

b. Write down a calculation for the volume of the tray if a square of any size is cut out.

c. Hence or otherwise determine the maximum volume of the tray, and the dimensions that correspond to this maximum value.

---

9. **A juice box**

A container to enclose 600 cm$^3$ of liquid is being designed in the shape of a square-based rectangular prism.

If the square base has x cm side lengths

a. Write down a calculation for the height of the container.

b. Write down a calculation for the total surface area of the container.

c. Find the dimensions of the container for which it has the smallest possible total surface area.

d. Make a scale drawing of the container with the dimensions found in part (c).

---

10. **A juice can**

Another container to enclose 600 cm$^3$ of liquid is being designed. This time a cylindrical shape is chosen.

If the cylinder has a base radius of r cm,

a. Write down a calculation for the height of the container.

b. Write down a calculation for the total surface area of the container.

c. Find the dimensions of the container for which it has the smallest possible total surface area.

d. Make a scale drawing of the container with the dimensions found in part (c).
11. **The ‘swing-wing’ fighter plane revisited**

Consider the swing-wing fighter plane as described in Question 4.

By letting \( x \) represent angle BAD,

a. find a (new) function for the area of triangle ACB in terms of \( x \).

b. Draw a graph of \( A \), the area of triangle ACB against \( x \).

c. Find the maximum value of \( A \) and the \( x \)-value for which it occurs.

12. **Cylinders within a sphere**

Consider all the possible cylinders that could be inscribed within a sphere of radius 10 units. Three examples are drawn below.

Out of the infinite possible cylinders, one has greater volume than all others. What are its dimensions? What proportion of the sphere’s volume does it enclose?

13. **Cones within a sphere**

Consider all the possible right cones that could be inscribed within a sphere with a radius of 10 units. Three examples are drawn below.

Out of the infinite possible right cones, one has greater volume than all others. What are its dimensions? What proportion of the sphere’s volume does it enclose?
14. **Cylinders within a cone**

Consider all the possible cylinders that could be inscribed within a right cone with a base radius of 5 units and a height of 10 units. Three examples are drawn below.

Out of the infinite possible cylinders, one has greater volume than all others. What are its dimensions? What proportion of the right cone’s volume does it enclose?
Answers

1. The cut straw
   a. i. 21 – 6
   b. i. \( \frac{1}{2} \times 6 \times (21 – 6) \)
   c. 5 \times \frac{1}{3} \times \left( \frac{40 - 4 \times 5}{2} \right)
   d. 45 cm²
   e. \( \frac{1}{2} \times (21 – x) \)

2. The pig pen
   a. i. \( \frac{40 - 4 \times 2.5}{2} \)
   b. \( \frac{40 - 4 \times 5}{2} \)
   c. \( \frac{25 \times 1}{3} \times \left( \frac{40 - 4 \times 2.5}{2} \right) \)
   d. \( \frac{25 \times 1}{3} \times \left( \frac{40 - 4 \times 5}{2} \right) \)
   e. \( 5 \times 1 \times \left( \frac{40 - 4 \times 5}{2} \right) \)
   f. \( 5 \times 1 \times \left( \frac{40 - 4 \times 5}{2} \right) \)
   g. \( \frac{1}{2} \times (21 – x) \)

3. The folded page
   a. i. \( \sqrt{21 - 5} \)
   b. \( \frac{1}{2} \times 5 \times \sqrt{(21 - 5)^2 - 5^2} \)
   e. \( \sqrt{(21 - x)^2 - x^2} \)
   d. \( \sqrt{(21 - x)^2 - x^2} \)
   f. 0 \leq x \leq 10
   g. \( x = 3.6 \text{ m or 6.4 m} \)
   h. \( x = 5 \text{ m} \)

4. The "swing-wing" fighter
   a. \( 2 \times \frac{1}{2} \times 3 \times \sqrt{10^2 - 3^2} \)
   c. \( x = 7.07 \text{ m} \)
   d. \( x = 7 \text{ m} \)
   e. \( x = 7 \text{ m} \)
   f. It looks as if the largest possible area is 50 m².

5. A graphical view
   d. \( x = 7 \text{ m} \)
   e. \( x = 7.07 \text{ m} \)
6. **The two pipe problem**
   a. $\sqrt{1^2 + 1.6^2 + \sqrt{(3 - 1)^2 + 1}}$
   b. $\sqrt{x^2 + 1.6^2 + \sqrt{(3 - x)^2 + 1}}$
   c. 

   \[ V = (1.6^2 + 1.6^2) \times (2 - x) \times \frac{1}{2} \]

   - 8. **The tin tray**
     a. The biggest is 30 cm$^2$
     b. The smallest is 0 cm$^2$.
     c. $x(60 - 2x)(90 - 2x)$
     d. Maximum Volume is $= 28520$ cm$^3$
     e. Dimensions are $11,771 \times 36,458 \times 66,458$

   9. **A Juice Box**
   a. $\frac{600}{x^2}$
   b. $2x^2 + 4x \times \frac{600}{x^2}$
   c. $8.434 \times 8.434 \times 8.434$
   d. $

   10. **A Juice Can**
   a. $\frac{600}{\pi^2}$
   b. $2\pi r^2 + 2\pi r \times \frac{600}{\pi^2}$
   c. radius is 4.571 cm
     height is 9.142 cm
     (diameter is 9.142 cm)
   d. 

   11. **Swing-wing fighter II**
   a. $\frac{1}{2} \times 2 \times 10\sin x \times 10\cos x$
   b. 
   c. Max. value of $A$ is 50 m$^2$
     and occurs when $x = 45^\circ$.

   12. A cylinder with radius of 8.165 and height of 5.773
     encloses 0.577 of the volume of the sphere.

     encloses 0.296 of the volume of the sphere.

     encloses 0.444 of the volume of the cone.
     (interesting ... ???)