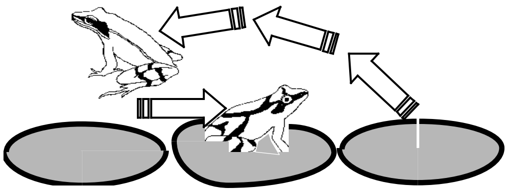
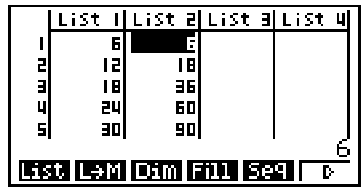
Describing and predicting the behaviour of systems

A unique learning experience – Teacher Notes





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Prince Alfred College

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CASIO

To the teacher

Introduction

This document may be of use to you and your students the way it is – or it may need alteration – minimal or extensive. It is used by Prince Alfred College as it is. Of course local rules mean that a one size fits all set of materials is almost impossible to write. When you read this document or work with it in a workshop I ask that you forget your local rules for a while and consider the journey from a mathematical point of view. Once you have done this I would suggest you put your local rules hat back on and see how you can make it work in your environment. We have tried hard to offer a unit that caters to both the less mature and extremely mature mathematical thinker. As the teacher you must know where to suggest a student leave a certain task and move on – we have not marker any part as 'harder' than the rest as I think this encourages students who think they can't to not. We believe that the materials lend themselves excellently to group work and encourage students to learn from each other – I have seen so many students that have minds that work better than me on materials like those enclosed – they don't need to know a lot, they just need to want to think.

It is **thinking** that is the 'holy grail" of this unit, not setting out or learning the 53 commandments or whatever – just thinking. It is the type of thinking that is encourages in this unit that we believe makes it so appealing form whatever and whoever's perspective you look and will ensure the students process the rest of the mathematics they encounter in a different manner than if they had not had this experience.

The approach

Remember 'fruit salad' algebra. Then forget it immediately.

The traditional approach to the teaching and learning of algebra (still seen in many learning materials) focuses on the student developing the ability to employ a series of rules and algorithms that are normally presented in a sterile manner that is not linked to anything that relates to the place where most students thinking is at the time. Maybe the worst part about this approach is that is does not provide the students with any idea of what the core ideas of algebra are about or why algebra is so important.

The *matchstick* revolution has done much to move the foundation algebra experiences of students to a better place. However, we are now in the year 2003 and the matchstick revolution really should be considered as 'traditional' – it is well over 20 years old. We now have a rather different mind set to all this and access to electronic technology that allows us to rethink the foundation experiences that are possible and *that should cause us to challenge the sequencing of the concepts* we know students need to develop.

Algebra is about describing the infinite number of possibilities that are possible for a given situation. How many students every get to actually realise this? How many think it is about manipulating letters in a subject they though was about numbers? The very first thing students experience in there algebraic development must be about the infinite set of possibilities before and how they can describe then succinctly and then work with the *description to reach some really simple but useful outcomes. Algebra is about making what seems impossible (working with the infinite) possible.*

The approach employed in this unit 'turns on its head' the traditional teaching approach to most of the areas of mathematics. It has been accepted practice to teach students the theoretical concepts and associated algorithms and then after they have mastered (what they probably don't really understand) ask the students to apply their new knowledge to some problem. We believe that if the learning of the theoretical concepts and associated algorithms is not driven by the need to learn it, as initiated by exposure to a problem that a student wants to solve, then a valuable opportunity to capture student interest has been lost.

It is possible to make the formality grow out of investigation. In fact we should be considering the investigation and problem solving endeavour as important a part of 'mathematics' as the rest.

It is hoped this unit of work will allow you to make sense of the dribble above and encourage you to believe that similar approaches are possible for *every* area of mathematics learning.

In this unit we have focussed on the investigation of "interest-rich/challenging" ***systems*** (activities and games at this level as the kids are little fellas), that require students to:

- interact with a *system* and determine the systems output in numerous ***specific situations***
- see/find/recognise and describe (*conjecture*) ***patterns*** in the systems output (behaviour)
- generalise patterns (the ***conjectures***) in ***words***
- generalise patterns (the ***conjectures***) using ***standard algebraic form***
- determine the ***underlying structure*** of a system that results in the behaviour
- ***prove*** that the conjectures are indeed true.

For the purpose of this unit we will define a system as:

a group of interacting parts that forms a whole.

The behaviour of the whole is what is initially studied to provide data from which a description of the behaviour may come. The study of the interaction of the parts is also important if one is to prove their conjectures about the wholes behaviour. It may be that some students do not progress through all of the above dot points. Fear not if this is the case, provided that your curriculum is structured so that students get another chance to traverse all the dot points at a later stage(s) when their personal level of maturity allows it.

This unit provides a ***foundation development*** of the ***key ideas of algebra*** (and a few more), it does not provide "text book" or "skill drill" work – that would inappropriate ***at this stage***. It does, however, provide the impetuous to proceed to what we call ***special studies***. The special studies are where students are asked to concentrate their efforts on building knowledge in a ***specific area of algebra*** (say linear functions) and to develop strong operative algebraic skills. Special studies are not necessarily approached in traditional text book ways, but tend to have some similarities.

Stenduser

The Stenduser (**St**arter-**En**der-**En**thuser): **The Number Neighbours Game** is designed to both raise the students interest/curiosity and to present a challenge which, for most, will require practice on the easier, graded activities that appear in the subsequent sections. It is an activity to have a go at, leave and come back to when you have some new knowledge. **Above all it is the students problem – not yours, so mask your lust, support them by answering their questions with questions to guide them but resist telling them the answers.**

You will need to 'work' the classroom well to ensure that the students understand the rules of the game. As you see a group do something wrong, use the approach of asking "Are you sure of that, I think you have broken one of the rules – look back at each rule again". Obviously you will treat each group a little differently.

Only those students who are "well on top of this activity" will succeed with Task 5. This task can be supported by asking the student, "Why can't you use only three counters in row 1?" The model response of, "2 can not be placed anywhere, as it is the 1st neigbour of both 1 and 3." Will indicate the students are in control.

The most important thing of all is to make students understand they this is 'kind of like' a Mt Everest problem (as are some that follow) and that each person will climb as far as they can. Different students will bail at different times. The idea of the personal Everest is really important and that you can come back any time to have another go as it does not go away. It is by using this idea that we believe these materials cleverly cater for mixed ability classes and/or group work. At the end of the day some will scale it in one go (not many), some will scale it on the first time they revisit and other may not quite summit without being carried – it is up to you if you carry them, but do not carry them to early.

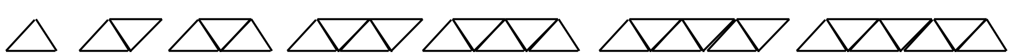
A possible alternative: Instead of using a board as we have you may like to use strings of beads. String 1 is to have no neighbours adjacent, String 2 to have no first or second neighbours adjacent, etc

2.1 Spot the Pattern – some answers

Note: **it is possible** to make an argument for **almost any answer** to questions of this "continue the sequence" type. Be wary of the "overly smart" from claiming they see repetitive patterns such as 22, 44, 66, 88, 22, 44, 66, 88, 22, 44, 66, 88, ..! Encourage lateral thought though.

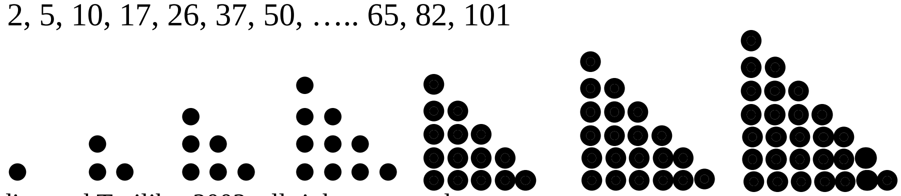
(a) 22, 44, 66, 88, …..110, 132, 154

(b)



(c) 2, 5, 10, 17, 26, 37, 50, ….. 65, 82, 101

(d)



(e) 1, 2,4, …..Several possibilities here!!
- 1, 2, 4, 7, 11, 16, ..
- 1, 2, 4, 8, 16, 32, ..
- 1, 2, 4, 8, 16, 31, (! really)

(f) S, M, T, W…..Thursday, Friday, Saturday, …

(g) 3, 1, 4, 1, 5, 9, ….easy as π

(h) 31, 28, 31, 30, 31, ….a calender sequence,
but 31, 28, 31, 30, 31, 28, 31, 26, 31, 24, …seems pretty reasonable too.

2.2 A fine nine

For example: 12345 × 9 = 12345 × (10 - 1) = 123450 - 12345 = 111105 = 111111 - 6

2.3 Mountain deep and valley high

There are 2^{n-1} valley creases, $2^{n-1}-1$ mountain creases and 2^n-1 total creases. Some students will simply add the two formulas of the mountains and valley to get a formula for the total and this opens up the opportunity to talk about equivalence again. These are readily seen by examining the patterns within the table, but they can also be deduced from the structure of the activity. The total number of creases is fairly easily seen, but the other two figures need some deductive skill. One could argue, for example, that at any stage after the first central valley fold half the paper is "face up" and half is "face down". Consequently each further fold creates equal numbers of valley and mountain creases. Since there are $2^n-1-1 = 2^n-2$ creases other than the central one, half of these are mountain creases (i.e. $(2^n-2)/2 = 2^{n-1}-1$ of them) and the other half are valley creases. Adding on 1 for the central valley crease gives a total of $2^{n-1}-1+1 = 2^{n-1}$ valley creases.
In Qn 5 working on a piece of A4 paper with thickness 0.1mm, after 20 folds the paper is almost 105m high – and less than 0.0003mm wide!

2.4 What do I have to do to this to get that?

This is maybe the most critical part of the unit. The problem that many teachers have commented about taking a patterning approach is that some student's (and teachers) just 'see the rule' and others just can't. After many years of watching and questioning those who 'can', we have written this section to help us all to be able to master the spotting of patterns. It is really just a matter of rapid-fire guess and check. So students need to have a good number awareness, be confident with numbers and the main operations we apply to numbers. They also need to be bold in what combinations they try. We have ***purposefully*** kept away from starting with linear patterns and then moving to quadratic and maybe in Year 10 showing them exponential patterns. Those 'boxes' are boxes that have been made over the years and, I feel, hold students back from exploring the operations and how they behave.

2.5 Ring, ring!

This is an activity where no conjecturing takes place and we have a human designed system that can be described algebraically of all possible cases.

The formula is $c = 40n + 25$.

Students should note that it is only useful here to substitute $30k$ for n.

5.37 minutes cost $5.05, a $9.85 call lasts more than 11.30 minutes and no more than 12.00 minutes.

2.6 The leaping ~~lizards~~ frogs

Answers: The total number of moves for n pairs of frogs is $n^2 + 2n = n(n + 2) = (n+1)^2 - 1$, again the notion of equivalent forms comes naturally from this.
$n^2 + 2n$, comprises n^2 jumps over other frogs and $2n$ sideways slides. One way to see this is that the total number of jumps must be n^2, each achieving a move of 2 units, while the total units to be moved is $2n(n+1)$, and hence $2n(n+1)- n^2= n^2 + 2n$.

With n red and m green frogs a similar argument gives the number of moves $=mn+m+n$.

2.7 Tower of Brahma/Hanoi

In this activity students will more than likely describe the patterns they see in a recursive manner. Don't discourage this, but show its difficult nature by asking then to find the 200 term in the sequence. Then encourage them to see if they can what we call a "link rule" or a formula connecting the two variables.

The following is from http://www.lhs.berkeley.edu/Java/Tower/towerhistory.html
The Tower of Hanoi (sometimes referred to as the Tower of Brahma or the End of the World Puzzle) was invented by the French mathematician, Edouard Lucas, in 1883. He was inspired by a legend that tells of a Hindu temple where the pyramid puzzle might have been used for the mental discipline of young priests. Legend says that at the beginning of time the priests in the temple were given a stack of 64 gold disks, each one a little smaller than the one beneath it. Their assignment was to transfer the 64 disks from one of the three poles to another, with one important proviso–a large disk could never be placed on top of a smaller one. The priests worked very efficiently, day and night. When they finished their work, the myth said, the temple would crumble into dust and the world would vanish.

Some useful on-line sites
http://www.askdrmath.com/dr.math/faq/faq.tower.hanoi.html
http://www.lhs.berkeley.edu/Java/Tower/Tower.html
http://www.pbs.org/teachersource/mathline/concepts/historyandmathematics/activity3.shtm

The world ends after $2^{64}-1$ days – approx 580 billion years
(cf age of Earth =1.2 billion years, age of man=2.4 million years!)

The task in Qn5&6 is essentially the inductive step in a proof by mathematical induction. This is a bit deep for some, but even if the students can make the structure link they are to be congratulated – remember groups are a great way to go here.

If you want a more directed form of Qn 5 [Tower of Brahma] try this

It may help to first consider the specific case of one more disc than you used in your experiments.
<u>Suppose that you checked the cases of 1 to 6 discs and consider the case of 7 discs.</u>

(a) What must be the situation before you can move the 7^{th} disc?

(b) How many moves are required to achieve this situation?

(c) How many moves to move the 7^{th} disc to a new pin?

(d) What is needed to complete the task?

(e) How many more moves are required to achieve this?

(f) Hence the total number of moves is …?

We have found dozens of web pages that run demonstrations of the Tower of Hanoi puzzle. This seems to be everyone's first java or JavaScript program to display on the web. Unfortunately, links often become invalid over time, and it is difficult to maintain a current list of nice demonstration programs. Here are two that particularly nice ones. We hope they still work when you read this.

Towers of Hanoi Puzzle http://chemeng.p.lodz.pl/zylla/games/hanoi5e.html
This rather nice interactive program by Romuald Zylla has the extra feature that you can have it scramble the disks into a random arrangement on the three pegs, giving the user more of a challenge. A hint button gives the next move in an optimal solution if the user is in doubt. The lazy user can hit the hint button repeatedly to see a full solution.

Multipeg Towers of Hanoi Game

This is a far more elaborate program, written by Xue-Miao Lu, one of the best researchers in the area of Tower of Hanoi variations. It can handle anywhere from 1 to 500 disks on any number of pegs from 3 to 30. Running in level 1, it demonstrates the "presumed optimal" algorithm for the multi-peg Tower of Hanoi problem; in level 4, the user can solve the problem by clicking on the pegs. Levels 2 and 3 allow the user to compete against the computer.

2.8 Exploring the behaviour of different formulas

This section is aimed at allowing students to get comfortable with formulas that look different and to see how they behave. The prime focus is to consider the effect of the adding of a squared term to a linear in a lead up to the next section. The use of technology is mandatory here if this is to be enjoyable for the students.

2.9 Whirlybirds

This activity is included because it is fun and serves to show students that not all systems are 'workoutable'. I have has serious physicists look at this problem and there conclusion is that the system is so complex it requires some high level physics to analyse it.

Students should drop the wb from the same height a number of times and average the results to ensure their data is as accurate as possible.

We chose to use the rather 'blackbox' nature of the technology here as we wanted the students to gain a formula quickly. In later units of work (later years) we 'pull apart' the internal goings on here. We also want the students to use their formula in the equations section.

Here is some data collected by students

l (cm)	2	4	6	8	10	12	14	16
F (sec)	1.3	1.8	2.1	2.3	2.2	1.9	1.6	1.2

2.10 The great Gauss

In (c), (d) note that the idea of representing an even number by $2n$ may need some discussion.

Part (e) shows that 2(1+2+3+4)=4 × 5

Part (f): the triangular numbers are of course indeterminate and so your diagram will contain some "…" symbols – which makes the proof less than totally convincing! But the algebraic version – the extension of Gauss's method in Part (g) – is a "good" proof.

In Part (g) young Gauss figured that his double count of the numbers from 1 to 100 is equal to 100(100+1) = 10100 and hence a single count would total 5050. Some versions of this story have Gauss at 11 years but either way this is a great story to "ham up" – students love to see one of their own outsmarting a teacher! (Although a student identifying with C.F.Gauss takes a bit of imagination!!)

It's worth practising the #terms×[average of (first & last)] version of this formula. It also works for any arithmetic series.

Example: 12+15+18+21+24 = 5(terms) × (average of 12 & 24 = middle term =18) = 90

(h)

$$\begin{array}{ccccccccccc} 5 & + & 10 & + & 15 & + & \ldots & + & 95 & + & 100 \\ 100 & + & 95 & + & 90 & + & \ldots & + & 10 & + & 5 \\ \hline 105 & + & 105 & + & 105 & + & \ldots & + & 105 & + & 105 \end{array}$$

Since there are 20 terms in the series the formula

(# terms)(average of first and last) gives $20\frac{(5+100)}{2} = 1050$

(i) The general formula for summing any n "equally spaced" numbers is

$$\text{Sum} = (\#\text{ terms}) \times (\text{average of first and last}) = n \times \frac{(F+L)}{2}$$

(j) Apply the formula to:

- 24 + 32 + 40 + 48 + 56 = 5(terms)×(average of 24 and 56) = 5×40 = 200
- 7 + 13 + 19 + 25 + …… + 1201 (200 terms) = 200× (average of 7 and 1201)
= 200×604
= 120800
- 15 + 19 + 23 + 27 + 31 + … + 215 has (215-15)/4+1 = 51 terms.
Hence Sum = 51(15+215)/2 = 6120

For some students this may be a struggle, so if you like the calculator offers an alternative:

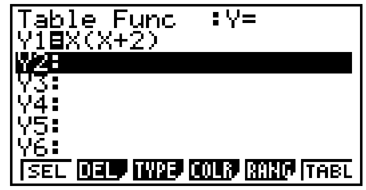
Return to the hopping frog puzzle. Suppose we want to know how many moves a player would have to make in order to play the game using from 1 to 100 pairs of frogs inclusive.

This means we would have to calculate the following SUM:

3 + 8 + 15 + 24 + _ _ _ _ _ + 10200

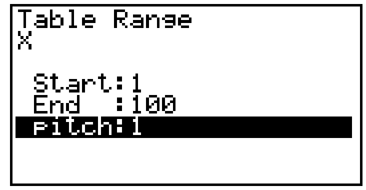
Follow the instructions below to see how the 9850GB PLUS will help to calculate this rather long SUM.

Enter the TABLE module and enter the link rule
y = x(x+2).



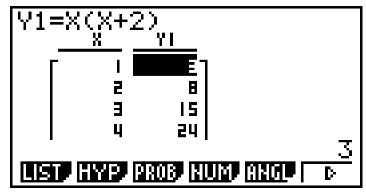
Access RANG (F5) and set the table range values as shown opposite.

Press the EXIT key.



Access TABL (F6)

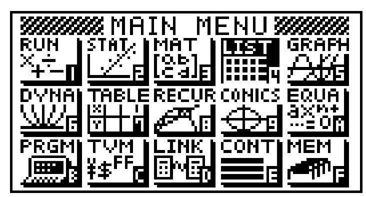
With the value highlighted asshown opposite press the OPTN key (stands for option) to reveal the menu seen opposite.



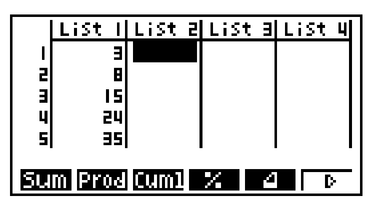
Access LIST, then LMEM and then List1.

This process will have copied the Y1 column into List 1.

Now press the MENU key and select the LIST module and press the blue EXE key.

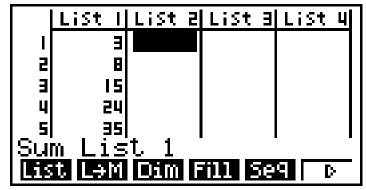


Press the OPTN key, access LIST (F1) and then access the arrow twice (F6 twice) to revela the menu seen opposite.



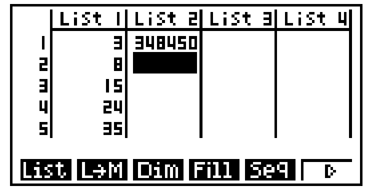
All Baker Centre work is considered as a WIP

Place the cursor as shown and
access Sum (F1),
then access the arrow (F6),
then List (F1) and
then press 1.



This will input the command as shown

Press the blue EXE key and the sum of the list will be shown.



Hence it would take a person 348450 moves to play the hopping frog puzzle with 1 to 100 pairs of frogs.

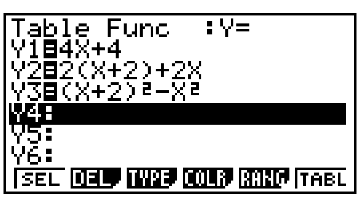
2.11 By the poolside

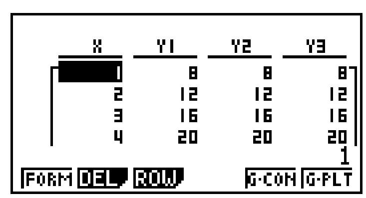
In this activity we are trying to really develop the students ideas so they can form strong ability to differentiate between a conjecture and a mathematical proof.

This is done by considering the approaches to reaching a formula – inductive logic (do a few specific cases, see a pattern and then guess the formula) as opposed to really pulling the system apart and understanding how it works – which is using deductive logic and hence ends in a formally proven result of which the truth is undeniable

For the square pool in Part (a) the possible formulas that students may spot include

- $N = 4n + 4$
- $N = 4(n + 1)$
- $N = 2(n + 2) + 2n$
- $N = (n + 2)^2 - n^2$





Verifying the formulas for any number of specific cases does not constitute a proof. Only the algebraic approach gives the proof of the result. **This is a great thing to put up on the walls of your room and re-visit when you get to the <u>special studies</u> to come later.**

For the rectangular pool in Part (c) the possible formulas that students may spot include

- $N = 2m + 2n + 4$
- $N = 2(n + m + 2)$
- $N = 2(n + 2) + 2m$
- $N = (m + 2)(n + 2) - mn$

2.12 Equivalent forms

This section is aimed at setting up the need to pursue formal algebra at a later stage and to introduce (formally) the ideas of inductive and deductive logic (which the students have been using through the whole unit – and before that!). It is hoped the students will start to gain an appreciation for the difference between seeing a pattern and guessing a formula and deducing the formula and hence knowing it works for all cases the system has to offer.

2.13 Do I really have to prove it beyond any doubt?

Self explanatory. Where the hell is that last region? The kids will be convinced the rule is the simple one that stands out – it is not, of course. This activity aims to reinforce the last one.

2.14 Chords and regions

- The pattern develops thus:

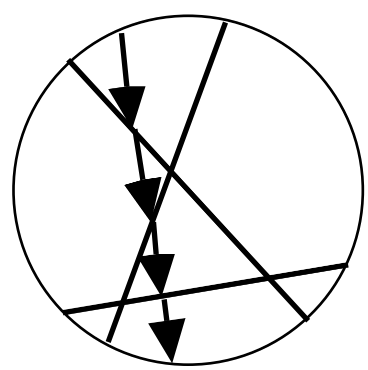
Chords	0	1	2	3	4	5
Regions	1	2	4	7	11	16
=	1	1+1	1+1+2	1+1+2+3	1+1+2+3+4	1+1+2+3+4+5

- The third row of the table shows that each additional level or triangle creates an extra "level number" of regions

- For the n th chord it appears that the total number of regions is

$$1+1+2+3+4+5+\ldots+n = 1 + n(n+1)/2$$

- Structurally we see that as we draw in the 4^{th} chord it intersects each of the previous 3 chords and then the far edge of the circle – and at each such intersection it creates an extra region: thus 4 extra regions are added to the previous number of regions.



  Similarly, the n^{th} chord intersects each of the previous $n - 1$ chords and then the far edge of the circle – and at each of those n intersections it creates an extra region: thus n extra regions are added to the previous number of regions.

  Hence the total number of regions for n chords is:
  the original 1 plus $1+2+3+4+5+\ldots+n = 1+1+2+3+4+5+\ldots+n = 1 + n(n+1)/2$.

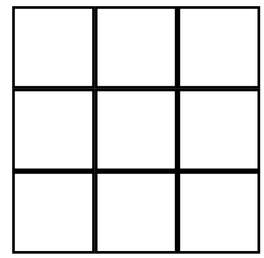
2.15 Squares in a square

An $n \times n$ square contains n^2 unit squares, $(n - 1)^2$ 2×2 squares etc, with the total number of squares of any size being

$$1 + 2 + 3 + 4 + \ldots + (n - 1)^2 + n^2 = (n/6)(n + 1)(2n + 1)$$

Students in junior secondary school will not usually know or be able to derive this formula, so it is reasonable to merely verify it here.

A 20×20 square would thus contain $(20/6) \times 21 \times 41 = 2870$ squares of any size.



2.16 Working with our formulas - equations

This is the icing on the cake section. Here the students see a use the formulas they have been generating and also have a very smooth introduction to what an equation is and how to solve it without all the algorithms to confuse them. It provides the impetus to do some special studies down the track on algorithms to solve equations.

The first question posed is an interesting one and it is hoped that *some* students will solve it intuitively by realising that $n(n+2)$ is almost n squared and just fiddle around with the numbers. The next one shows that is not so easy in some instances, but also shows that those students who could see the 'intuitive' way can still proceed.

2.17. Justin, Laura and Hulu - some problems to solve

ANSWERS – E&OE ! – These are for students who are on top of their game.

1. Assume Justin stands to the left of Laura and then double the total possibilities to allow for Laura being left of Justin.
- If J&L have 18 people between them, there's just one place J can be (extreme left)
- If J&L have 17 people between them, there's two places J can be (extreme left or 2nd left)
- If J&L have 16 people between them, there's three places J can be (extreme left, 2nd or 3rd left)

….etc
- If J&L have 3 people between them, there's 16 places J can be (positions left 1-16)

Hence total possibilities = 2(1+2+3+4+5+….+16) = 272

2. (a) 4 (b) 9 (c) 16 (d) 25 (e) $50^2 = 2500$ (f) n^2

3. • 4 + 10 + 18 + 28 + 40 + 54 + …= (4) + (4+6) + (4+6+10) + (4+6+10+12) + … suggests that the 80th term with be 2 less than the sum of the first 81 even numbers – i.e. 81(2+162)/2 - 2 = 6640

• Similarly the general formula for the *n*th term is (n + 1)(2 + 2(n + 1))/2-2, which could be simplified to $t_n = n^2 + 3n$. Alternatively some students might see this just by examining the pattern of numbers. However it is obtained it could be entered in the TABLE on the calculator and then transferred to the LISTs before finding the cumulative sum (use the method described in Section 5.1). This shows that **a total of 183600 guests arrived**.

• More advanced students may be able to get this result algebraically:

$$\sum_{n=1}^{80} n^2 + 3n = \sum_{n=1}^{80} n^2 + 3\sum_{n=1}^{80} n = \frac{80 \times 81 \times 161}{6} + 3 \times 80 \times \frac{1+80}{2} = 183600$$

4. The triangular numbers are illustrated in Section 2.1(d) and Section 2.4.

(a) The 20th term is 20(1 + 20)/2 = 210

(b) The 200th term is 200(1 + 200)/2 = 20100

(c) With 2 points we get 1 line, with 3 points we get 3 lines, with 4 points we get 6 lines, etc and it appears that we are getting the sequence of triangular numbers. And hence with *n* points we get

$$1 + 2 + 3 + \ldots + n = n(n + 1)/2 \text{ lines}$$

[there are of course a number of alternative ways of deducing this result, but this method seems in keeping with the approach developed in this unit.]

(d) With 80 points we get 80(1 + 80)/2 = 3240 lines.

5. This is essentially an easier version of Qn 3 above and same methods can be applied. The *n*th lowering of the drawbridge admits $5n$ - 2 people and summing the first 50 of these gives a total of 6275 arrivals – leaving **3725 guests yet to come**. Examining the cumulative sums in the LISTs on the calculator shows that the last two groups to enter are Group 63 with 318 guests and Group 64 with the last 46 guests – thus breaking the pattern.