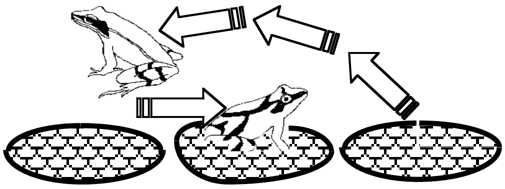
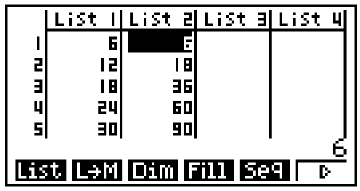
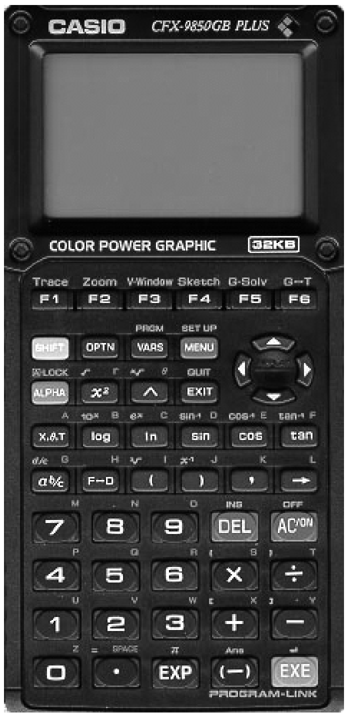
Describing and predicting the behaviour of systems

A unique learning experience – Student Booklet







A product of the Noel Baker Centre for School Mathematics

WIP (Work in progress)

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Section **Page**

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1. Stenduser: Number Neighbours

Warm up task

If you don't have counters for this activity you can cut up the grid of numbers supplied at the back of this booklet.

Given counters numbered 1 to 10, arrange them so that no two consecutive numbers are next to each other.

Given counters numbered 1 to 8, arrange them so that no two consecutive numbers are next to each other.

Given counters numbered 1 to 6, arrange them so that no two consecutive numbers are next to each other.

Introduction

In this activity we will use only the set of numbers we call the positive integers, that is

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, ...

Now consider the concept of a '*number's first neighbour*'.

The number 5 has two **first neighbours**, 4 and 6.
4 is called the *lower first neighbour* and 6 is called the *upper first neighbour*.

The number 24 also has two **first neighbours**, 23 and 25.

Now consider the concept of a '*number's second neighbour*'.

The number 5 has two **second neighbours**, 3 and 7.
The number 24 also has two **second neighbours**, 22 and 26.

Third neighbours, fourth neighbours, fifth neighbours and so on exist.

Task 1

Complete the following table for the number **28**:

Neighbour number	**1st**	**2nd**	**3rd**	**4th**	**5th**	***k* th**
Lower neighbour #						
Upper neighbour #						

Task 2

- Using the counters numbered 1 to 10 arrange them so that no number is next to ***either*** of it's first neighbours ***or*** either of its second neighbours.

- Using the counters numbered 1 to 8 arrange them so that no number is next to either of it's first neighbours or either of its second neighbours.

Task 3 - Now for a game – the rules.

This game is played on a board consisting of an grid of squares never ending in both the horizontal and vertical directions, we could say it is a k by k where $k=\{1,2,3,4,\ldots\}$.

For practical reasons not ***all*** of the squares are shown on the board provided! Note that the board is supplied separately.

Your task is to place the **minimum possible number of counters** in the rows of the game board according to the following rules:

- Each row must contain more than one counter

- Squares in each row must be filled from the left edge, with no spaces between counters

- If a row contains n counters then counters numbered 1 to n must be used.

- In Row 1 no number may be next to either of its first number neighbours.

- In Row 2 no number may be next to its first number neighbours **nor** its second number neighbours

- An equivalent rule applies to Row 3, Row 4, …. Row k.

Form a team of three students and

- **Play the game using only Rows 1 to 3 (inclusive)**
- **Minimise the number of counters used in each row.**
- **Document your results below, writing down the number of counters used in each row and the total – also put your result on the whiteboard.**

Row Number	Your minimum number used
1	
2	
3	
Total	

Task 4 – Refining your strategy

If your teams score is not as low as the lowest produced by another team (or the teacher, play again and refine your strategy until you can achieve the lowest number of counters.

Task 5 – Predicting behaviour – for specific cases

- Predict the minimum number of counters required for row 10 and write down how the counters can be positioned. Explain how you did this.
- Predict the minimum **total** possible for rows 1 – 10 inclusive
- Predict the minimum number of counters required for row 50 and write down how the counters can be positioned. Explain how you did this.
- Predict the minimum **total** possible for rows 1 – 50 inclusive
- Suggest two formulas, one that would generate the minimum number of counters required for row k and one that would generate the minimum **total** possible for rows 1 – k inclusive

Task 6 – Analysis the system

This game can be considered as a system.
We will define a system as **a group of interacting parts that forms a whole.**

- When you are fairly convinced that you have mastered the minimisation of the number of counters, explain in simple logical manner, why the number of counters you have used in ***each row*** is the minimum number possible.
- Determine a way of explaining how to position the counters so that the rules are not broken

Task 7 – Proving your conjectures correct – for an infinite number of cases (in general)

- Use your output from Task 6 to write down a logical argument (a proof) that shows your first conjecture from Task 5 is correct – beyond any doubt.
- Prove that your second conjecture from Task 5 is also correct – beyond any doubt.

Task 8 – Finishing off

- For which row is the minimum number of counters 200?
- Is it possible for a row to use exactly 2650 counters as a minimum value?

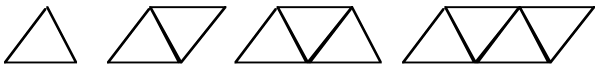
The End

2. *The Learning Journey*

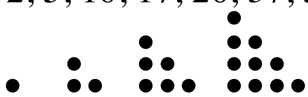
2.1 Spot the Pattern

Give the next three terms in each of these patterns. In each case state the "rule" or "structure" of the pattern that you have seen.

(a) 22, 44, 66, 88, …..

(b) 

(c) 2, 5, 10, 17, 26, 37, 50, …..

(d) 

(e) 1, 2, 4, …..

(f) S, M, T, W…..

(g) 3, 1, 4, 1, 5, 9, ….

(h) 31, 28, 31, 30, 31, ….

2.2 A Fine Nine

Consider this number pattern:

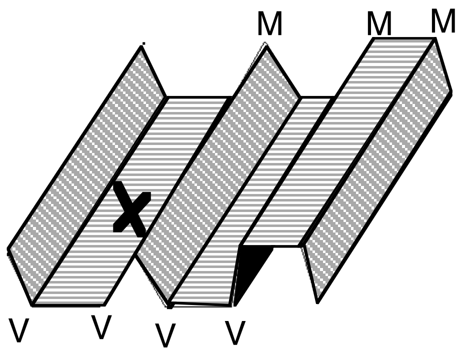
$$123 \times 9 + 4 = 1111$$
$$1234 \times 9 + 5 = 11111$$
$$12345 \times 9 + 6 = 111111$$

(a) Extend the pattern for three rows in each direction.

(b) Think about how this system works and try to explain why it behaves the way it does?
[Hint: think of an alternate, indirect way of multiplying by 9]

2.3 Mountain deep and valley high

Take an A4 piece of paper (or an A3 if you have one). Mark one face with a cross to denote this to be the uppermost face. Lay it on the table (cross upwards) and fold it in half going from left to right. Be sure to crease the fold well. Open the paper so it is A4 sized again. It has only one crease line and that has formed a "valley". We will call this a valley crease. Returned the paper to the 'folded in half' position and fold it in half again. Open the paper so it is A-4 sized again. Notice this time that it has more valley creases but also some creases that form 'mountains' - we will call these mountain creases. Your job is to continue to fold in halves and keep track of the number of valley and mountain creases. Record your findings in the table below.



The number of folds will vary – and so we will define all the possible values as *f* and the number of valley crease associated with *f* to be *V* and we will use *M* to denote the varying number of mountain creases and *T* for the total.

number of folds (*f*)	1	2	3	4	5	6
# of valley creases (***V***)						
# of mountain creases (***M***)						
total number of creases (***T***)						

2. Write down all the patterns you can see in the table of numbers

3. (a) Use your data to conjecture a rule that links *V* and *f*

(b) Use your data to conjecture a rule that links *M* and *f*

(c) Use your data to conjecture a rule that links *T* and *f*

4. Use your **'link rules'** to **predict** how many of each type of crease will be present if the paper is folded
(a) 10 times
(b) (b) 20 times

5. Analyse the workings of this paper folding system and see if you can mount a logical argument that proves your conjectures true for all cases.

6. If you could stand on top of your piece of A4 paper which has been folded 20 times, would you fit under a 3m ceiling? And how wide would the folded paper be?

2.4 What do I do to this to get that?

You probably found it fairly easy to find the *recursive* pattern in the paper investigation, ie multiply the previous term by 2 and then add one ($\times 2+1$).

Finding the link rule probably provided a much greater challenge. The human mind seems more naturally attuned to seeing how the next term is related to the previous rather than how one quantity is linked to another. Also, the **combination of operations** is often more simple in the recursive pattern than it is in the link rule.

To identify **link rules** at this stage it really is a case of trial and error and being prepared to **try** lots of different combinations of operations in trying to answer the question,

"What do I do to this to get that?"

Some people's minds can operate with numbers faster than others and are also more flexible in what they will try, and so they can **guess and check** pretty rapidly.

To be successful all you need to do is understand the following operations:

$$+,-,\times,\div,(\ )^n$$

The activities that follow are aimed to improve you ability to try lots of different **combinations of operations** to meet a certain end point. **Be daring in what you try out**.

Example

a) What do I do to 1 to get 1?

There are a multitude of answers to this.

- $+0$
- $\times 7-6$
- square it
- cube it
- $\times(200010-200000)\div 10$ (is $\times 200010-200000\div 10$ the same)

b) What do I do to **1 to get 1** that will satisfy the question, what do I do to **2 to get 3**? A table may help to view this:

n	1	2
M	1	3

There is a also more than one answer here

- clearly $\times 7-6$ does not work anymore
- $\times 2-1$ works, as does
- multiplying the first number by one more than itself and then halving the result or putting it in symbols $\frac{1}{2}n(n+1)$
- finding the *n*th power of 2 and subtracting 1 also works (2^n-1)

c) What do I do to **1 to get 1** that will satisfy the question, what do I do to **2 to get 3** that will also satisfy the question **3 to get 7**?

A table may help to view this:

n	1	2	3
M	1	3	7

There is a also more than one answer here

- $\times 2-1$ **does not work anymore**
- nor does $\frac{1}{2}n(n+1)$
- (2^n-1) still works
- but so does squaring the number, subtracting the number from this and then adding 1 (n^2-n+1)

So which is correct for the paper folding? Try a few more cases out and see. The fact is, to prove which is correct requires some logical thought, some deductive argument, at this point we are just guessing and checking, being inductive, not deductive. Your teacher will help you to be deductive with the Paper Folding activity later on. At this stage stay with the guessing and checking.

You should have realised BEDMAS is really important here – if you need to brush up on this seek some materials from your teacher to do so.

In the following tasks ***remember to be daring (your teacher will provide a chocolate frog for daring results).***

Task One – The birth year game

Write down the year in which you were born. Using all the digits in this number and the operations $+, -, \times, \div, (\ )^n, \sqrt{\ }, \sqrt[n]{\ }$ (ignore the last two if you like) in some combination with the digits make the numbers 1 – 20 inclusive.

Eg. I was born in 1963, so I could make 1 with 1+9-(6+3)

Task Two

a) Write down five different ways that 3 can be operated on to get 11
b) Write down five different ways that 3 can be operated on to get 157

c) i) What do I do to **1 to get 2**?
ii) What do I do to **1 to get 2** that will satisfy the question, what do I do to **2 to get 6**?
iii) What do I do to **1 to get 2** that will satisfy the question, what do I do to **2 to get 6** *that will also satisfy* the question, what do I do to **3 to get 12**?
A table may help to view this:

n	1	2	3
M	2	6	8

d) i) What do I do to **1 to get 2**?
ii) What do I do to **1 to get 2** that will satisfy the question, what do I do to **2 to get 5**?
iii) What do I do to **1 to get 2** that will satisfy the question, what do I do to **2 to get 5** *that will also satisfy* the question, what do I do to **3 to get 10**?
A table may help to view this:

n	1	2	3
M	2	5	10

iv) if n was 15, what do you think M might be?
v) if n was n, what do you think M might be?

e) i) What do I do to **1 to get 2**?
ii) What do I do to **1 to get 2** that will satisfy the question, what do I do to **2 to get 7**?
iii) What do I do to **1 to get 2** that will satisfy the question, what do I do to **2 to get 7** *that will also satisfy* the question, what do I do to **3 to get 24** *that will also satisfy* the question, what do I do to **4 to get 77**?
A table may help to view this:

n	1	2	3	4
M	2	7	24	77

vi) if n was 11, what do you think M might be?
vii) if n was n, what do you think M might be?

2.5 Ring ring!

My mobile phone company charges me 40¢ per 30 second block of time plus a 25¢ "flagfall" or connection charge for each call made. If any part of a 30 second block is used, you are charged for the whole block.

1. Calculate the cost of a calls lasting for 30s, 60s, 90s, and so on. Document your work in a table.

2. If my call uses *n* 30-second blocks write a formula which links the cost, *c* (cents), *n*. This formula describes the cost variation of all possible lengths of phone calls.

3. Check that your formula produces a cost of $3.85 for a phone call of 4.20 minutes.

4. Calculate the cost of a call lasting for 5.37 minutes (5 minutes 37 seconds)

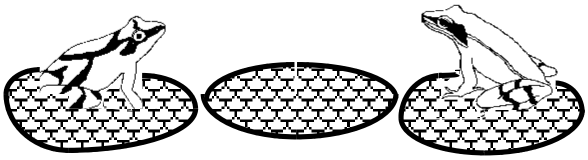
5. One call that I made cost me $9.85. What can you say about the length of that call (other than it was too long!)?

Note that you have not made any conjectures here, but formulated your formula from the structure of the system.

Return to Stenduser?

2.5 Leaping ~~Lizards~~ Frogs

When walking through the park last weekend I observed some tiny frogs playing a game similar to leap frog on some water lily pads. The game went as follows :

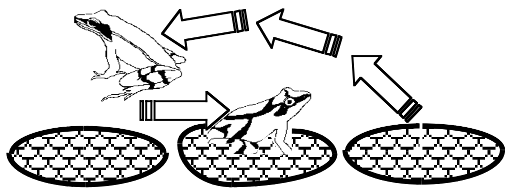


GF		RF

A green frog sat on one lily pad and a red frog sat on another **with an empty lily pad in between.**

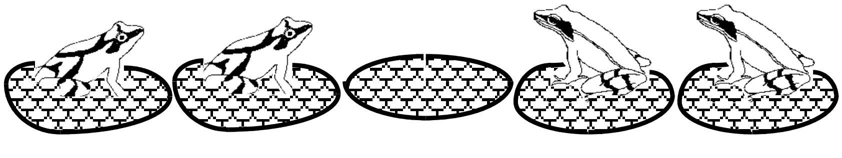
The aim of the game was for the frogs to swap places by either:

- **sliding onto an empty pad,** or by
- **jumping over a frog on the next lily pad onto a vacant lily pad.**



Can you see that it takes just 3 moves (either slides or jumps) to complete the game?

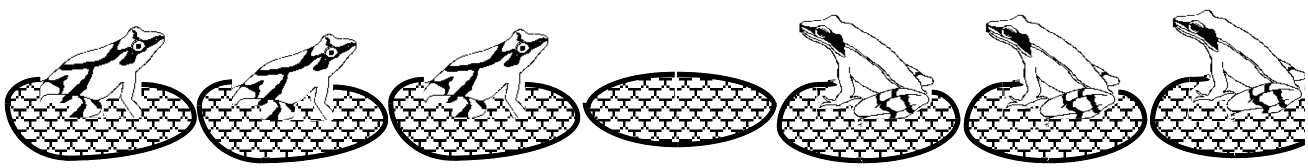
After successfully completing the game each frog called over a friend of the same colour to sit on a lily pad by their side as shown below. Now there are 2 pairs of frogs (i.e. 2 green frogs and 2 red frogs) playing the game. As always there was only one lily pad in between the red and green frogs at the beginning.



GF	GF		RF	RF

Once again the frogs either **slide onto an empty pad** or **jumped over a frog onto a vacant lily pad** until the green frogs were at the right end of the row of lily pads and all the red frogs at the left end.

After each game was over the frogs kept on calling one more frog of each colour to join them for the next game.



1. Play the game with the counters and board supplied to you by the teacher. Start with one pair of frogs and then go to two and so on. Draw up a table in which you can record the number of pairs of frogs (n) in the game and the minimum number of moves (M) required to complete the game.

2. Write down all the patterns you can see in the table of numbers

3. Use your data to conjecture a 'link rule' that links M and n

4. Use your 'link rules' to **predict** the minimum number of moves required with
   (a) 6 pairs of frogs
   (b) 20 pairs of frogs

5. Analyse the workings of this frog system and see if you can mount a logical argument that proves, beyond any doubt, your conjectures true for all cases.

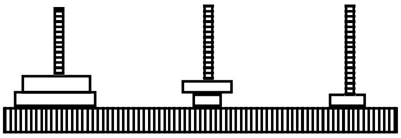
6. What happens if there are unequal numbers of red and green frogs?

2.6 The Tower of Brahma

According to some legends, at the beginning of time the gods presented the monks of Brahma in the Indian city of Benares with 64 fragile gold discs – all of different sizes and stacked in decreasing order of size on one of three diamond studded pins.

The monks were instructed to move the stack of discs to another pin in the minimum number of moves, according to the following **rules**:

- only one disc may be moved per day
- no disc may be placed on top of a smaller disc



This would be an illegal move

It is said that when the monks complete their task the temple will crumble into dust and the world will vanish in a clap of thunder.

How long to the end of the world?

You should have learned by now that the way to start any investigation is to try a simple form of the problem firstly to see that you fully understand the rules and then to collect some data (a 'table' of results) in hope of seeing a pattern emerge. Later we may like to analyse the system to see if we can understand why it behaves the way it does.

1. Obtain a less expensive version of the discs and pins, with about 6 discs.

2. Experiment to find the **least number of moves** for each number of discs.
[As a check you should find that 3 discs require 7 moves.]

Number of discs (*n*)	**1**	**2**	**3**	**4**	**5**	**6**
Minimum number of moves (*M*)						

2. Write down all the patterns you can see in the table of numbers

3. Describe in words how the number of moves required is related to the number of discs.

A note on shorthand notation:

- a^3 means $a \times a \times a$

  e.g. $4^3 = 4 \times 4 \times 4 = 64$

- a^n means $a \times a \times^x a \ldots \times a$

  (n of these)

  e.g. $3^2\ 3^n = (3\times3) \times (3\times3\times\ldots\times3)$

  $= 3^{(n+2)}$

4. Write a formula that connects the number of moves M with the number of discs, n.

5. Can you be sure that your formula is correct – that is that it will **always work**, that is for any number of disks – which is in fact and infinite number of disks?

   For example if you have found the number of moves for 6 discs (by trial and error) can you work out how the system works by thinking about the "**movement structure**" of the game to figure out how many moves are needed for 7 discs? Does this check with your formula, and can you obtain the result using the game discs?

6. Use the method of Question 5 to construct an argument that proves that if your formula is true for n discs then it will also be true for $n + 1$ discs.

7. **How long to the end of the world?**

2.7 Exploring the behaviour of different formulas

So far you have seen a number of different formulas that describe the behaviour of some interesting systems.

In the simplest sense a **formula** is just **a set of numbers linked by operations**. For example:

$$n(n + 2)$$

is really **some number *multiplied* by 2 more than that number**

$$40n + 25$$

is really **some number *multiplied* by 40 with 25 *added* on**

$$2^{n-1}-1$$

is really **the (*n*-1)th power of 2 subtract 1.**

So, if a formula is just a collection of numbers linked by operation it might be interesting to see how certain types of formulas behave. By **behave**, we mean **how does the output of the formula vary when the value inputted changes.**

Example

Let see how the formula $\boldsymbol{y=5x + 2}$ behaves. We could be really haphazard here and just try any input values for x, but it is more useful to choose a block of values, say $0 \le x \le 5$ where x is an integer.

Note that we have chosen y and x to denote the quantities that vary as we don't have a context like number of frogs or number of moves or whatever.

To see the behaviour we can make a table as follows:

x	1	2	3	4	5	6	7	8	9	10
y	7	12	17							

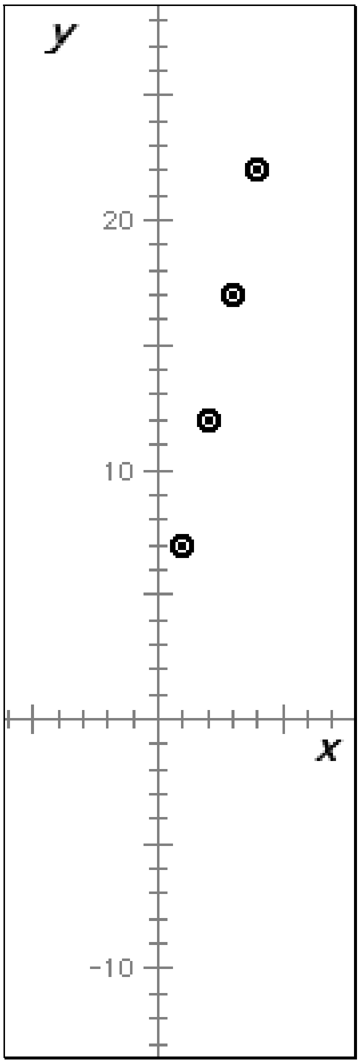
The arithmetic involved in this formula is so simply **you should do it in your head**. Complete the table.

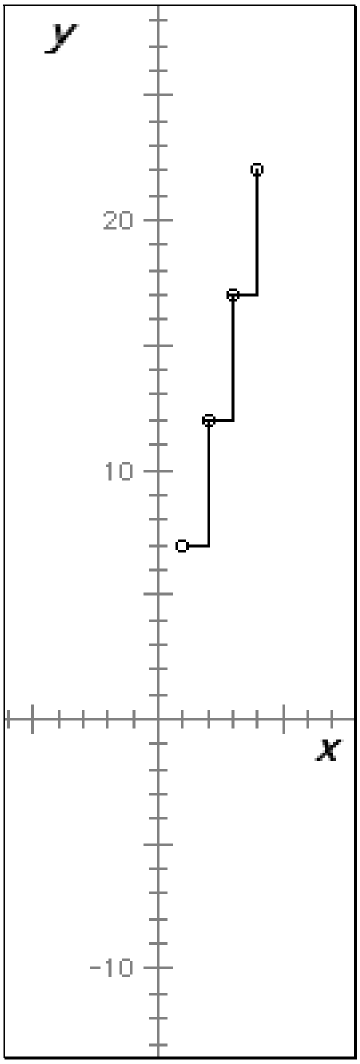
You should see that this formula behaves in a very interesting manner – **the next *y* value is 5 greater than the previous**.

Not only does a table of numbers help us to see the behaviour of a formula, but a graph can as well. A graph is an alternative representation of the formula and shows its behaviour in a different way.

You can see that if the x and y value are plotted as a point (like on a street directory) it shows that the **points fall in a straight line** which is the same as saying as **the *x* value increases by one, the *y* value increases by a constant amount (5 in this case).** The second graph shows the **constant steps.**

These types of graphs are called **scatter plots**





1. For each of the following formulas **produce a table of values** for a block of integer *x* values of your choice, (you must have at least 6 *x* values) and **draw a scatter plot** to represent the output of the formula. Write down any things of interest you observe.

   a) $y = 3x + 2$
   b) $y = 1x + 2$
   c) $y = 10x + 2$

Adding an x^2

The formulas you looked at above were all of the same kind – they have a fancy name – **linear functions** (we will worry about that later).

Here is a questions to ponder – **'How would happen if we added on the square of a number -an x^2 to $3x + 2$ so that it became $x^2 + 3x + 2$?'**

Before you proceed, discuss how you think this will behave with the class.

To see the behaviour we can make a table as follows:

x	1	2	3	4	5	6	7	8	9	10
y	6	8								

As a class complete table and draw a scatter plot of the output. Discuss anything interesting you find. How has the addition of the x^2 changed the behaviour?

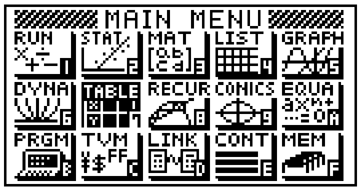
2. For each of the following formulas **produce a table of values** for a block of integer x values of your choice, (you must have at least 6 x values) and **draw a scatter plot** to represent the output of the formula. Write down any things of interest you observe.

   d) $y = x^2 + 3x + 2$
   e) $y = x^2 + 6x + 2$
   f) $y = x^2 + 9x + 2$

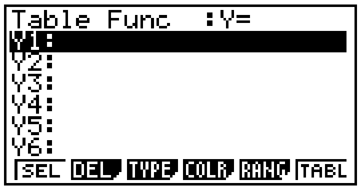
Using some technology to help

The point of the exploration is not to practice your mental arithmetic (it is important to practice that though). It is to see how changing the formula by adding different operations affects the behaviour of the output. Graphics calculators or spreadsheets are wonderful at doing lots of computation quickly and allowing us to see the output in both table and scatter plot form.

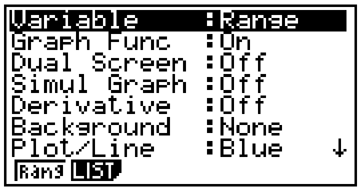
With the calculator turned on and the main menu visible, use the arrow key to highlight the TABLE menu.



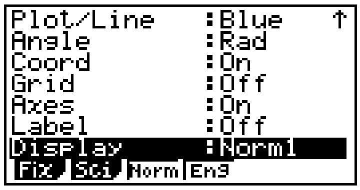
Then press the blue EXE key (alternatively, simply press 7). The following screen will result:



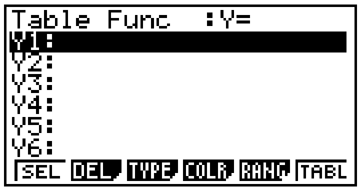
Now press SHIFT and then MENU to reveal the 'SETUP' screen for this module. Set each option as shown right. Use your arrow keys to scroll down to the 'hidden options'.



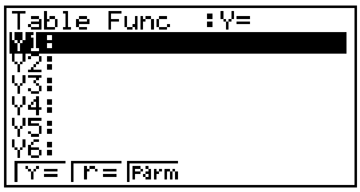
You can experiment with different settings later.



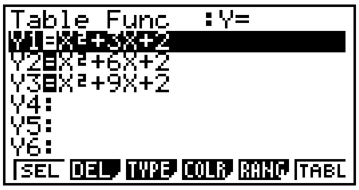
Press the black EXIT key to return.



If your screen does not have Y1 and so on, on the left, use TYPE (F3) and then Y= (F1) to make it so.



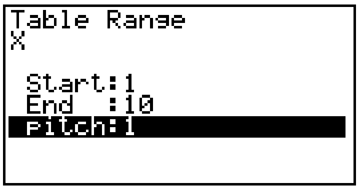
We will now enter the LINK RULES we explored above



Press the X , θ , T key, then the *x2* key then + then 3 then the X , θ , T key, then + and then 2 , to enter $y = x^2 + 3x + 2$

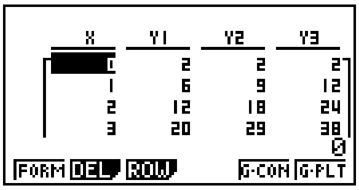
Carry out a similar procedure to enter the two other link rules.

We now need to tell the calculator the x values that we wish it to use to calculate y values.



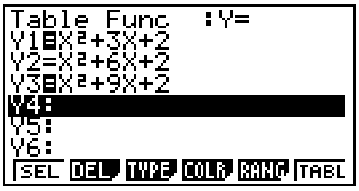
Use RANGE (F5) and set the values of start end and pitch to those shown opposite. Pitch is the incremental jumps that you wish to have in the table.

Press the EXIT key when you are done.

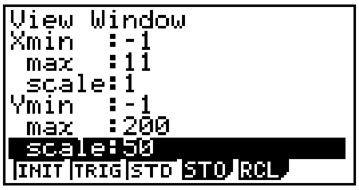


Use TABL (F6) to produce the table.

You can navigate the table using the arrow keys.



Should you **not** want to have all the link rules appear in the graph, press EXIT, use the arrow keys to select the rule(s) you do not want (for this exercise select Y2) and use SEL (F1). The = sign will no longer be surrounded with a dark rectangle and is said to be NOT SELECTED.

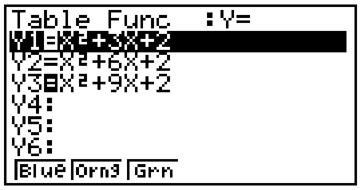


If we wish to draw a graph that illustrates each of these link rules we must first set the scale of the axes – called the **view window** on this calculator.

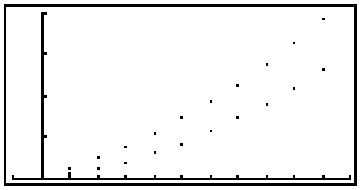
Press SHIFT and use V-WIN and the View Window settings will appear.

Set the values as shown opposite, **think about why we have set them this way.**
Press the EXIT key

We can now set each of the link rules to have different coloured graphs. Selected the first rule, use `COLR (F4)` and then choose the colour you want by pressing the appropriate key, either `F1`, `F2` or `F3`. Simply arrow down to select the other rules and choose a colour. Press the `EXIT` key.
Now use TABL(F6) to re-draw the table and then use G-PLT to produce the required graph.

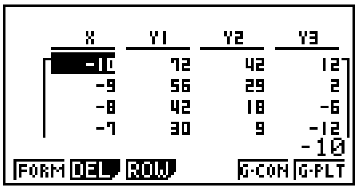


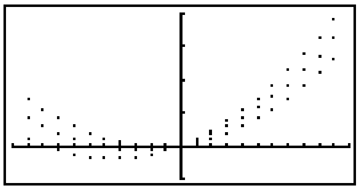
Not seen in colour here unfortunately.



The beauty about this technology is that you can go back and change the RANGE of the table and hence explore the formula's behaviour for different inputs very quickly.

The screen shots below show the behaviour of the three rules compared to each other for $-10 \leq x \leq 10$, I looked very closely at the table of values before I set up the view window. See if you can re-create the screen shoots below.





3. Use your graphic calculator to explore the behaviour of the following formula pairs and from this discuss the effect of adding an $\boldsymbol{x^2}$ or an $\boldsymbol{-x^2}$ term to something like $5x + 2$.

   a) $y = -3x + 2$ and $y = x^2 - 3x + 2$
   b) $y = -5x + 2$ and $y = x^2 - 3x + 2$
   c) $y = x^2 - 3x + 2$ and $y = -x^2 - 3x + 2$

3. Use your graphic calculator to explore the behaviour of the following and from this discuss the effect of adding an $\boldsymbol{x^3}$ term to something like $x^2 - 3x + 2$

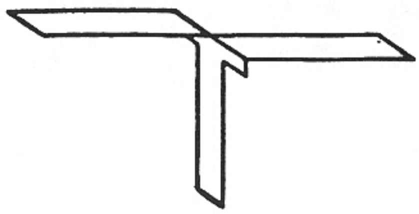
   a) $y = x^2 - 3x + 2$ and $y = x^3 + x^2 - 3x + 2$
   b) $y = x^2$ and $y = x^3 + x^2$

4. Explore some things for yourself – document your work.

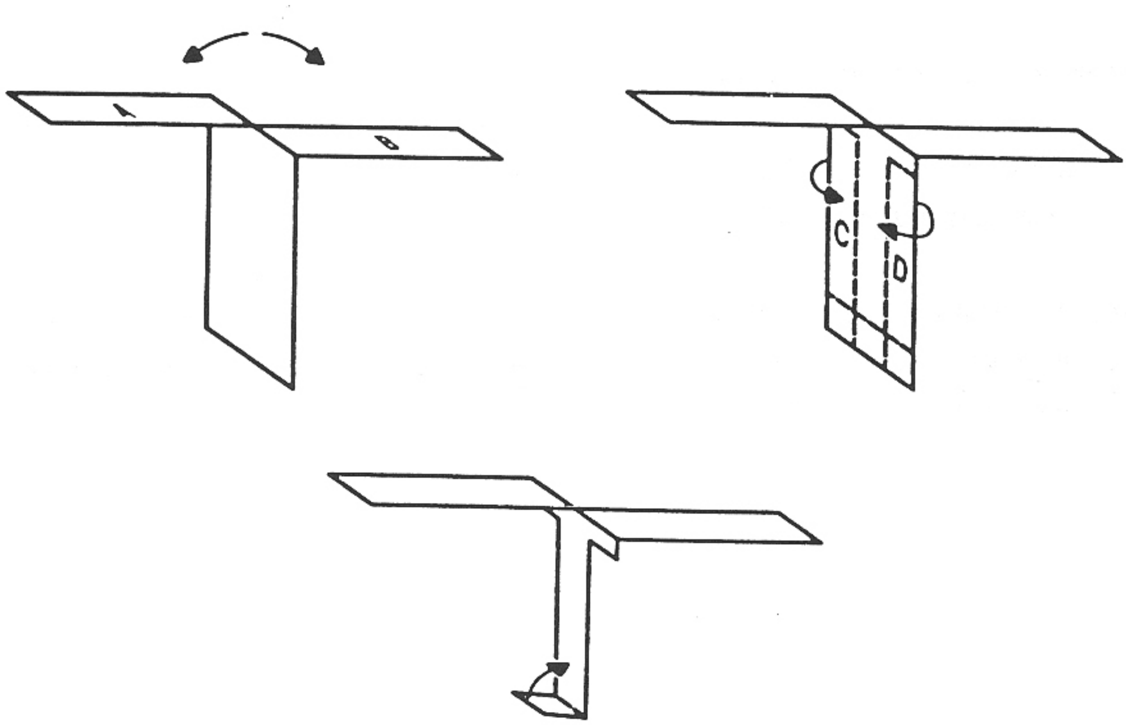
2.9 Whirlybirds – they are real spinners

The inner workings of some systems are really quite difficult to analyse. In such cases we can only rely upon the data we collect from observing the system in action and then attempt to find a formula that seems to be a suitable predictor of the behaviour. It may be virtually impossible to prove that the formula we decide on is 'correct', but if it seems to predict the behaviour reasonably well then we can use it, but with caution.

So what in the world is a whirlybird? It is flying device that looks something like this:



When you drop it, it flies to earth in the fashion of a helicopter. How do you make one? Use the template provided at the back of this booklet and fold as shown below.



Use a paper clip to hold the base of the bird together.

Without altering the bird in any way, find a second story balcony or equivalent and give your bird a test flight.

1. **Your aim is to investigate how the flight time (*F*) of the bird from release point to ground varies as you change the length (*l*) of the bird.**

   Work in pairs or threes and release your bird from the same height each time, but each time 'clip his wings' a little. **You may like to use the table that follows as a guide. You may need to think a lot about how you proceed so that you get a good set of data.**

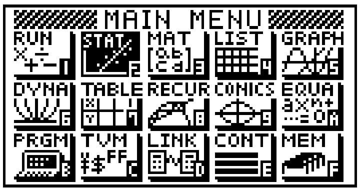
l (cm)	2	4	6	8	10	12	14	16
F (sec)								

Working out a formula

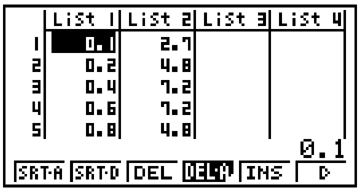
The table below has some data that was collected from a **totally different system** – nothing to do with a whirlybird.

x	1	2	3	4	5	6	7	8
y	5.1	8.7	13.5	18.0	23.4	27.6	33.4	37.2

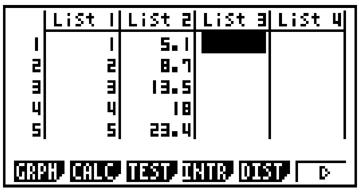
Enter the STAT mode of your calculator



If you have data already in the lists and you don't want it then delete it as follows. Put the cursor in the list somewhere, use F6 to reveal more screen buttons and then use DEL−A to delete all data in that list. Press YES (F1) when prompted
Repeat as necessary.



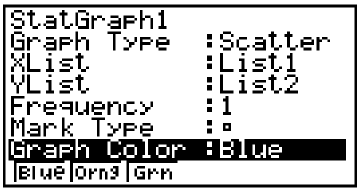
Press F^ to return to the starting list of screen buttons and then enter the *x* values in List 1, pressing the blue EXE key between entries and the values in List 1 and the *y* values in List 2.



Can you see what the formula should be from just looking at the data?

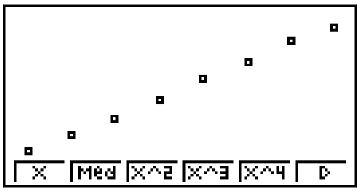
To get a graphical representation of this data, set the *view window* as appropriate.

Then use GRPH (F1) and then SET (F4) to set up the type of graph we want the machine to draw. Make the settings as seen opposite. We want a scatter plot as we have done before.



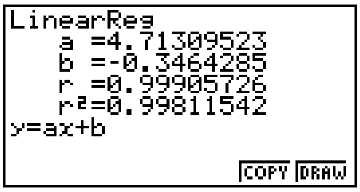
Press EXIT and then use GPH 1 (F1) to draw the graph.

Look familiar?



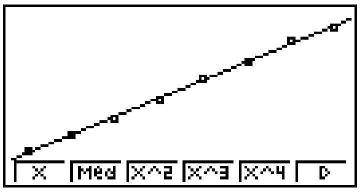
Now this sort of technology has the ability to workout a formula given some data. It will give you the best formula it can based on the data – and it will give you lost of different types of formulas. You have seen some of these already.

Use X (F1) and see what happens.



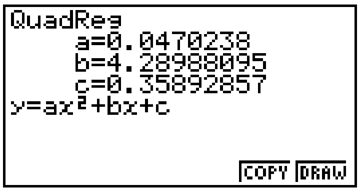
This tells us that if we want a formula that looks something like $y = ax + b$ then the one that most closely predicts the data is $y = 4.71x - 0.35$

Use the DRAW (F6) option and see what happens.



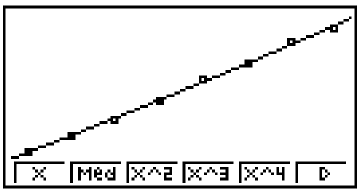
The calculator draws a line, all the points on this line are generated by the formula it thinks is the best one – it allows you to see if the formula is OK.

Press EXIT and then use GPH 1 to draw the graph again. This time press use X^2 (F3) and see what happens.



This tells us that if we want a formula that looks something like $y = ax^2 + bx + c$ then the one that most closely predicts the data is $y = 0.047x^2 + 4.29x + 0.359$

Use DRAW (F6) again to see what the formula does – seems pretty good.



How do you know which is best?
If you can analyse the internals of the system that can be a help, sometimes it is just very obvious which is better and sometimes it is not – like this case here.

2. Use your calculator to find a formula for your whirlybird system. First try a formula of the type $y = ax + b$ and then one of the type $y = ax^2 + bx + c$ – which seems better? Why?

3. Write down the formula you think is the better of the two you will need it later. Compare it to other people in your class – if it differs try to work out why.

4. Try some of the other formula types offer by the calculator – are any better than others?

2.10 The Great Gauss

Consider the number pattern:

$$2+4+6+8 = 4\times 5$$
$$2+4+6+8+10 = 5\times 6$$
$$2+4+6+8+10+12 = 6\times 7$$

1. Check that it is valid and then if it is extend the pattern for three rows in each direction.

(b) Use your pattern to find the following sums, explain how you did it:

- 2+4+6+……+198+200

- 1+2+3+…….+99+100

(c) Generalise these results by completing the following ***conjectures***.

- the sum of the first n even numbers is ….

- the sum of the first n natural numbers is ….

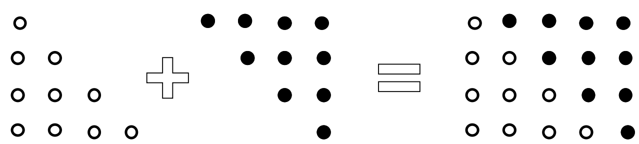
(d) Write the following conjectures using symbols

- 2+4+6+……+2(n -1)+2n = …..

- 1+2+3+…….+(n -1)+n = …

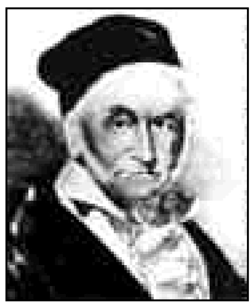
> **A note on notation:**
> - $2a$ means $2\times a$
> $a(a-1)$ means $a\times(a-1)$

(e) Which of the specific results above is demonstrated by the following diagram?



(f) Draw a diagram to extend the argument in part (e) above to the general result: $2+4+6+\ldots.+2n = n(n+1)$

(g) Carl Friedrich Gauss (1777-1855) is one of the greatest mathematicians of all time. Many stories are told about his arithmetic skills when he was very young. One relates how a 3 year old Carl found an error in his father's account book! Another tells how at age 6 he was pestering his school teacher with far too many "why is that so?" questions and so, to buy time to pay attention to his other students, the teacher sent young Gauss off to "add up all the numbers from 1 to a 100". Unfortunately for the teacher, Gauss was back in less than a



minute – and with the right answer! When asked to show his work, Gauss muttered something about having added the numbers up twice and showed the teacher his slate with just the following on it:

```
  1  +  2  +  3  + . . +  99  + 100
100  + 99  + 98  + . . +   2  +   1
```

- Explain the internal workings of this little system, and state the answer?
- How does this relate to the "proofs" you developed in parts (e), (f) above?
- Extend Gauss's method to prove the general case for 1+2+3+…+n

This result is very useful in a range of mathematical problems and applications and is well worth remembering. Write it here and in your book.

1 + 2 + 3 + 4 + …… + *n* =

(h) Use the method of young Gauss to add the multiples of 5 from 5 to 100, i.e. 5 + 10 + 15 + …. + 95 + 100

(i) Generalise the method to give a formula for summing any n "equally spaced" numbers.

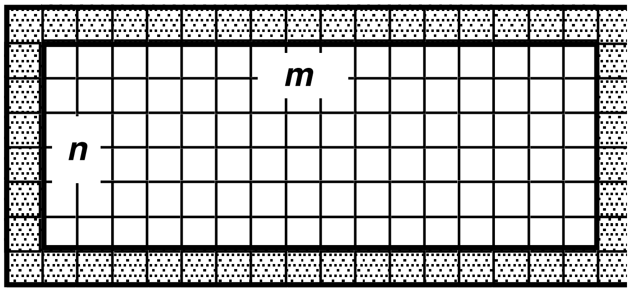
(j) Apply your formula to find the sums:

- 24 + 32 + 40 + 48 + 56
- 7 + 13 + 19 + 25 + …… + 1201 (200 terms)
- 15 + 19 + 23 + 27 + 31 + … + 215

Return to Stenduser?

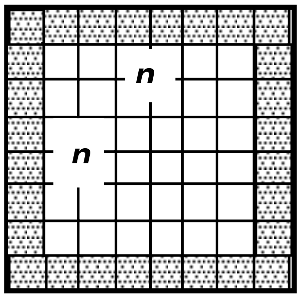
2.11 By the poolside

Picture this – a nicely tiled rectangular garden pool with a tiled border which is one tile wide. If the pool is *m* tiles long by *n* tiles wide, how many border tiles are there?



(a) Before we answer this problem let's consider a simpler one of an *n* by *n* square pool. Find a rule or "formula" for the number of border squares by either:

- **looking at a sequence of specific cases and looking for the patterns, and/or**
- **by considering the way that this system actually works**



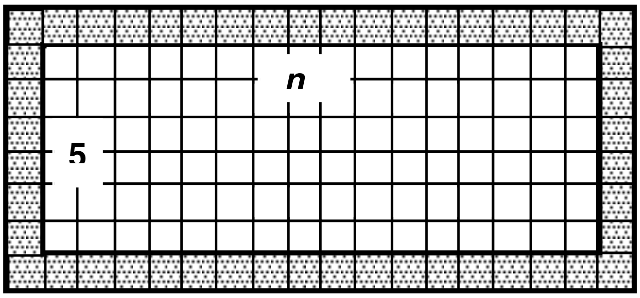
A note on notation:
$2(a+2)$ means 2 lots of $(a+2)$
or $(a+2)$ lots of 2
or a lots of 2 + 2 lots of 2
or $2a+4$

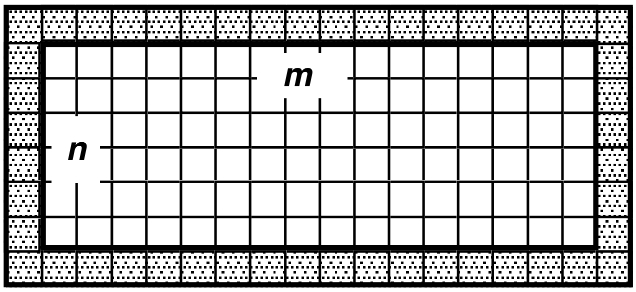
Write your answer on the white - board – is it the same as the formulas found by other people?

(b) Can you show that all the other formulas are "**equivalent**" to yours?
- Test each formula using n = 1, 2, 3
- Does this prove the formulas are equivalent? Explain your answer.
- **Use the table function on your calculator to test each formula using *n* = 1 to 100**
- Does this prove the formulas are equivalent?

(c) Which approach was necessary in part a) if you were to develop a formula that you know will work **in all cases** (or in general)? If you used the first approach what have you developed?

(d) Now consider pools for which the width is fixed at 5 tiles wide. How many border tiles are needed to surround a pool which is $n \times 5$? Find another formula for the number of border tiles are the two formulas equivalent.





(e) Now find a formula for the number of border tiles needed for an $m \times n$ pool.

(f) Have you arrived at a conjecture or a formula **about which there is no doubt** will work in all cases? Explain your answer.

2.12 Equivalent forms

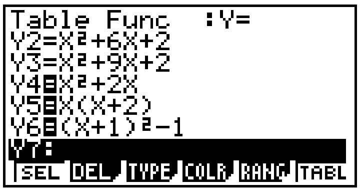
In the activities you have dome thus far you should have seen it is possible for people to come up with different formulas that seem to produce the same output. Can they both be correct? Are they, in fact, the same formula or equivalent formulas?

Think back to the frog hopping investigation, you saw the following different formulas

$$M = n^2 + 2n$$
$$M = n(n + 2)$$
$$M = (n+1)^2 - 1$$

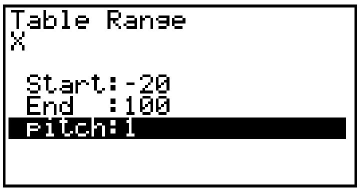
How can we test to see if these are equivalent? We could substitute LOT and LOTS of values for *n* and see if they give the same output for *M*.

We will use our graphic calculator to do this.

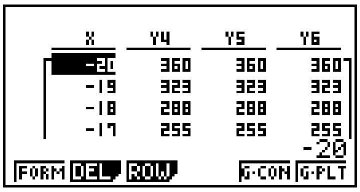


Enter the formulas as shown before in TABLE mode. Do not delete the other formulas you have entered, just deselect them by putting the cursor on them and using SEL (F1).

In this case *M* is Y4 and so on and *n* is X.



Now set the tables range using RANG (F5) so that lots and lots of values are used.



Now press EXIT and use TABL (F6) to produce the table. It may take a while – if you asked the machine for too many it may even tell you MEM ERROR – we all have our limitations.

Explore the table and see what you think – unfortunately however, **this does NOT prove beyond any doubt** that these formulas are equivalent. There just may be **one** value for *n* that the formulas produce a different output for.

To prove it beyond any doubt, you would need to try every possible value – and that would be very laborious. Or would it? Do you think it would be possible?

Proof beyond any doubt is what mathematics different to all other sciences. Proof beyond any doubt is based on logical argument (or **deductive reasoning**). Guessing a formula from seeing a pattern is an example of what is called **inductive reasoning**. This does not give proof beyond any doubt.

You have seen some of this already. You may have to wait a while to see how to prove that the above formulas are equivalent.

1. a) Check that the pairs of rules below are equivalent rules for $-10 \leq p \leq 100$ where p is an integer

   i) K = p(p+3) and K = $p^2 + 3p$
   ii) K = p(p+5) and K = $p^2 + 5p$
   iii) K = p(p+10) and K = $p^2 + 10p$

   b) Also check that the rules below are equivalent for
   p = -3,-2.5,-2,-1.5, _ _ _, 30

   i) K = p(p+3) and K = $p^2 + 3p$
   ii) K = p(p+5) and K = $p^2 + 5p$
   iii) K = p(p+10) and K = $p^2 + 10p$

   c) Use what you have found in a) to **induce** what you think will be the equivalent
   rule for

$$K = p(p+q)$$

2. a) Check that the pairs of rules below are equivalent rules for $-10 \leq p \leq 100$ where p is an integer

   i) K = $2^{p+3}.2^4 - 1$ and K = $2^{p+7} - 1$
   ii) K = $2^{p+3}.2^p - p$ and K = $2^{2p+3} - p$
   iii) K = $2^{2p-1}.2^{1-p} - p^2$ and K = $2^p - p^2$

   b) Also check that the rules below are equivalent for
   p = -10,-9.5,-9,-8.5, _ _ _, 20

   i) K = $2^{p+3}.2^4 - 1$ and K = $2^{p+7} - 1$
   ii) K = $2^{p+3}.2^p - p$ and K = $2^{2p+3} - p$
   iii) K = $2^{2p-1}.2^{1-p} - p^2$ and K = $2^p - p^2$

   c) Use what you have found in a) to **induce** what you think will be the equivalent rule for

$$K = 2^a.2^b - v$$

3. a) Determine which of the of rules below you think is equivalent to the rule $K = (p+4)(p-4)$

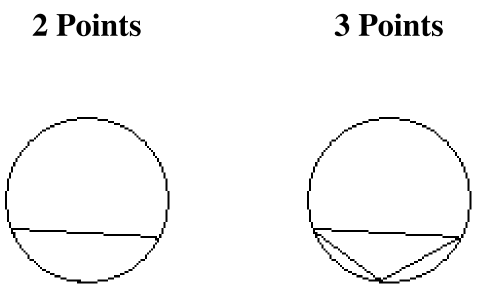
i) $K = p - 4$
ii) $K = p^2 - 16$
iii) $K = p^2 - 4$

b) For how many values of p would you have to show a rule does not match with its counterpart to conclude that it is not identical. Why?

2.13 Do we really have to prove it beyond any doubt?

Consider the following system. Take a circle and put five points around the circumference and join each point to each of the other points with a line segment. **You must do this so that no three of the line segments fall on the same point in space** – in other words any intersection point must be formed from **only two line segments**.

Below you can see what happens with 2 and 3 points.



For each different number of points (*n*), count up the number of regions (***R***) that the circle is divided into.

1. Complete the following table, ***if at any point you think you can see a formula for this system then complete the table without drawing and counting.***

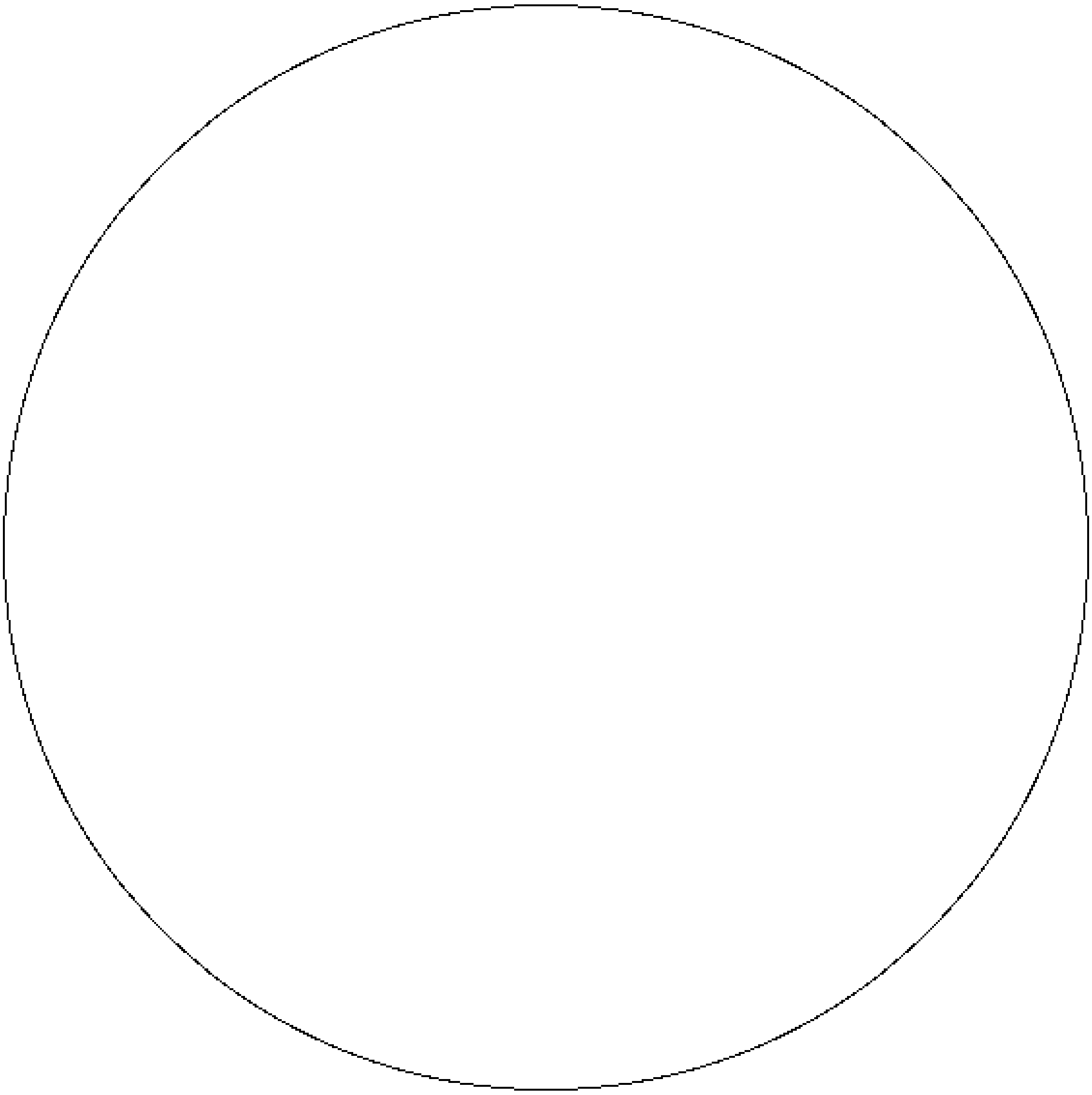
n	2	3	4	5	6	7	8
R							

2. On the circle supplied on the next page, draw 5 points on the circumference and join as requested. Count the number of regions you get. Does it match with the formula you thought was correct?

3. Now test the case when $n = 6$

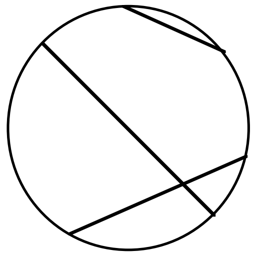
4. Can you came up with a better formula? Maybe you might like to use your calculator to help.

Do you now appreciate why we need to prove – if it is at all possible? See if you can analyse the workings of this system.

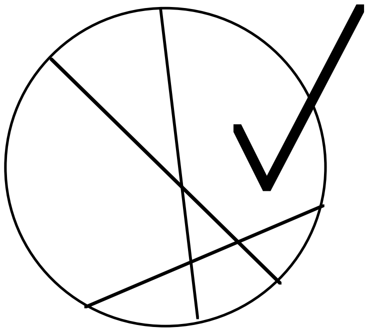


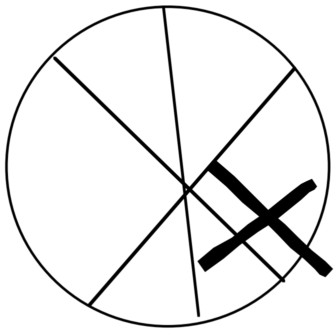
2.14 Chords and regions

A **chord** of a circle is a line segment joining any two points on a circle. The diagram at right shows a circle with three chords drawn in.



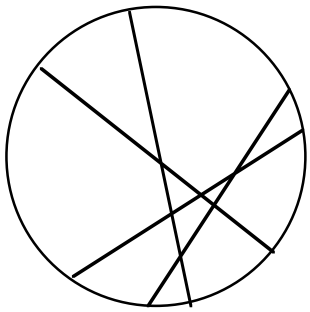
Consider a circle in which each chord intersects each other chord so that exactly 2 chords intersect at any one point – like the last problem.

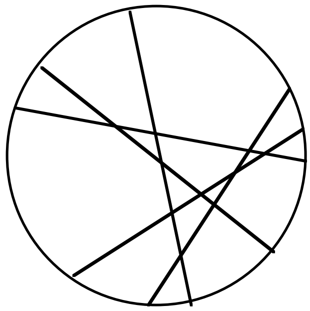




Notice that the circle with 3 chords the circles has been divided into **7 regions**.

What happens to the number of regions as the number chords varies?





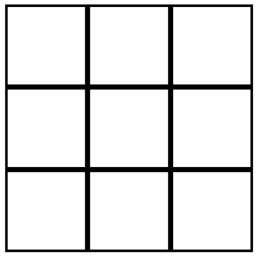
1. If *n* denotes the number of chords and *R* denotes the number of regions, construct a table that shows the number of regions for 1 to 6 chords inclusive.
(You may need to draw a larger diagram for the case $n = 6$.)

2. Show that the rule $R = \frac{1}{2}n^2 + \frac{1}{2}n + 1$ generates the correct *R* values for $n = 1$ to 6 inclusive.

3. Show that $R = \frac{n(n+1)}{2} + 1$ would **seem** to be an equivalent rule for the one given in 2. Explain what you did.

2.15 Squares in a square

In the diagram at right there are nine squares - right?



Actually there are fourteen! Can you see them all?

1. **Investigate the pattern** in the number of squares, *N*, contained in squares of varying sizes where *n* is the number of squares on one side of the big square. Document your work in a table.

2. Show that the following formulas are equivalent and generate the values of that you found in part 1.

$$N = \frac{1}{6}n^3 + \frac{1}{2}n^2 + \frac{1}{3}n \quad \text{and} \quad N = \frac{n(n+1)(2n+1)}{6}$$

3. Use your calculator to find the number of squares that **may** be found in a 20×20 square. Explain what you did.

2.16 Working with our formulas - equations

Lets recap some of the formula's we have generated throughout this unit.

Frogs hopping about: $M = n(n + 2)$

Towers of Brahma: $M = 2^n - 1$

Whirlybird: $F = -0.02l^2 + 0.35l + 0.71$ (yours will differ to this but should be of the same form).

1. Answering the following question and explain clearly how you did it. There are many more than one method.

   How many pairs of frogs were used if the minimum number of moves was 251000?

2. Answering the following question and explain clearly how you did it.

   If the whirlybird took exactly 2 seconds to fall, how long was its wings?

One way to proceed with the questions asked on the last page is to think in the following manner.

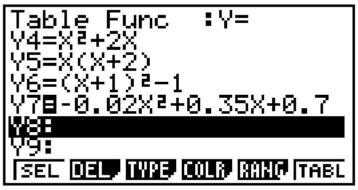
If there are 251000 moves then *M* is equal to 251000, so we could say:

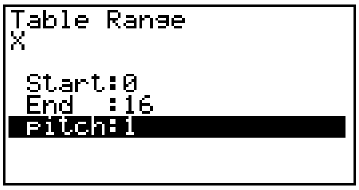
$$251000 = -0.02x^2 + 0.35l + 0.71$$

When the lead part of a formula is replaced with a number in this way the result is called and ***equation***. All we need to do is find the value of that returns 251000 when substituted in for *l*.

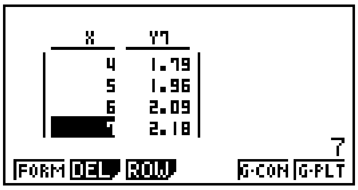
Using our calculator here we can do as follows:

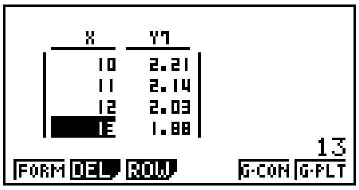
Enter the formula into table mode as shown before and then set the range of the table to around what you think may be correct – I have chicken out here – look at my range.





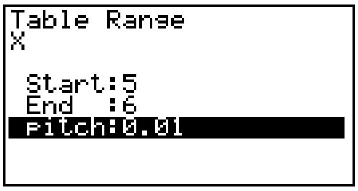
Producing the table and exploring it carefully gives:

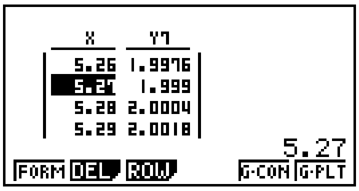




We can see it seems that two different wing lengths will provide what we want. One that is between 5 and 6 centimetres and one that is between 12 and 13 centimetres.

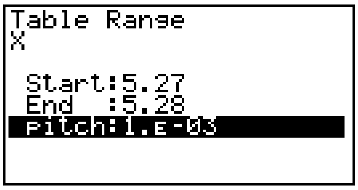
To find a close approximation to this value, we can reset the tables range. First for the one between 5 and 6.

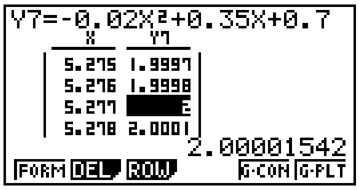




So, we find some wing length between 5.27 and 5.28 centimetres will do.

We could get even more accurate:





GOT IT – or did we? Not that if you put the cursor on the number 2, it is not really 2 at all!!

Just how close do we need to get? How accurate can you cut that wing?

So we can say that if $M = 5.277$ the time of flight would be close to 2 seconds.

We call 5.277 a *solution* to the equation $251000 = -0.02x^2 + 0.35l + 0.71$ and say that we have *solved the equation*.

1. Find the other solution to the equation $251000 = -0.02x^2 + 0.35l + 0.71$

2. a) Write down an equation that will help us to find how many pairs of frogs were used if 17688 moves were require.
   c) Solve the equation

3. While playing the Towers of Brahma one Saturday afternoon I found it took me a minimum of 4095 moves. Write down an equation that will help you to find the number of disks I had and then solve it.

4. Recall exercise 2.13 – Chords and Circles. Write down an equation that will help to find the number of chords required if there are 904 regions and solve it.

5. Recall exercise 2.13 – Chords and Circles. Is it possible if there are 3251 regions? Explain your answer.

6. Find as many solutions as you can to the equation:

$$x^3 - 2x^2 - 5x + 6 = 1$$

7. Jack claimed he took 23420 moves when playing the frog game and he is sure that was the minimum value for the number of pairs of frogs he had. Is his claim true or false? Explain your answer.

Return to Stenduser?

2.17 Some problems to solve

1. Justin and Laura are two members of a 20 strong youth group. If the group are asked to stand in a straight line, determine in how many positions Justin and Laura stand so that there are three or more people between them.

2. (a) Find the sum of the first 2 odd numbers
(b) Find the sum of the first 3 odd numbers
(c) Find the sum of the first 4 odd numbers
(d) Find the sum of the first 5 odd numbers
(e) Use the CASIO 9850GB Plus to find the sum of the first 50 odd numbers
(f) Write down a formula for the sum of the first n odd numbers

3. Last night my door bell rang 80 times. The first time it rang 4 guests arrived, the second time 10 guests arrived, then 18 then 28 and so on. How many guests did I receive?

4. Consider the sequence 1, 3, 6, 10, ….. This is called a sequence of **triangular** numbers – why?.

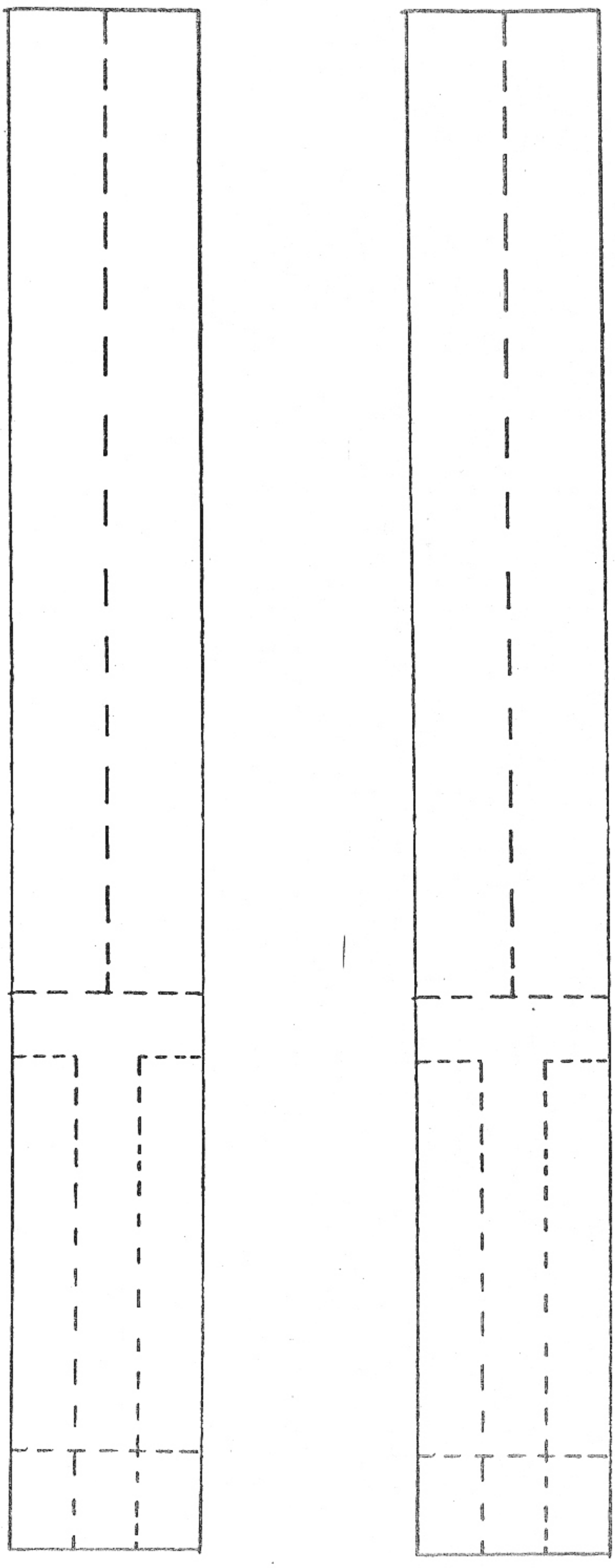
(a) Find the 20th term in this sequence.
(b) Find the 200th term in this sequence.
(c) Given a set of n points in a plane, no three which are collinear, determine how many line segments could be formed.
(d) Given a set of 80 points in a plane, no three which are collinear, determine how many line segments could be formed.

5. The king of Nanastam decided to have a reception for the marriage of his daugther Hulu to the Prince of the Northland. Many invitations were sent out far and wide, in all 10 000 guest were invited. When the quests arrived for the grand occasion they did so in a strange way. The first lowering of the draw bridge saw three people cross it. The next time eight people entered, the next time thirteen entered and so on.

When the drawbridge had been lowered fifty times, the king asked the guard at the bridge how many guests were still to come. What should have his answer been?

If all of the 10 000 guests attended and entry continued as explained, determine if the number of guests in last group to pass the drawbridge violated the pattern.

6. Ask your teacher to supply you with some competition problems to which you can apply your new (and powerful) skills.



1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
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1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**