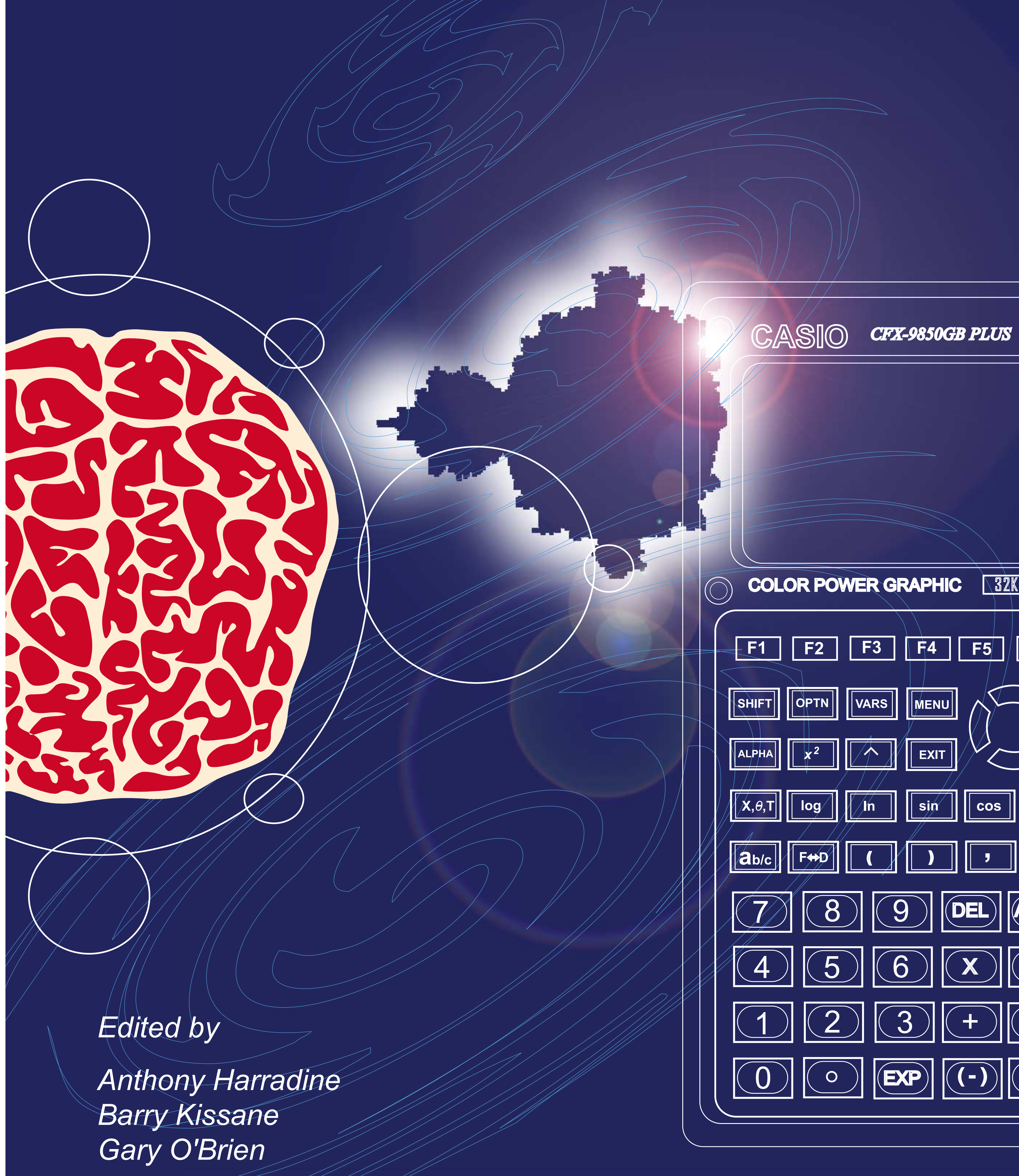
Enhancing Learning with a Graphics Calculator



Edited by

Anthony Harradine
Barry Kissane
Gary O'Brien

Enhancing Learning With a Graphics Calculator

Enhancing Learning With a Graphics Calculator

First published in Australia in 2002
by Shriro Australia Pty Limited
A.C.N. 28 002 386 129

Shriro Australia Pty Limited
23-27 Chaplin Drive, Lane Cove NSW 2066, Australia
Telephone: 02 9415 5052
Facsimile: 02 9415 5057
Email: casio.edusupport@shriro.com.au
Internet: http://www.casio.edu.shriro.com.au

Cover concept by Anthony Harradine
Cover design and production by Scott Parker
Printed by Falcon Print, South Australia

Enhancing Learning With a Graphics Calculator
ISBN 0 9578302 1 1

CONTENTS

FOREWORD

In recognition of the need for professional support, the Casio Education Division invited Australian teachers to submit lesson ideas that they would like to share with their colleagues. This publication is a collection of these lessons that will provide teachers with more practical ideas for the use of graphics calculator technology in the classroom. This book comprises nineteen lessons that all teachers can use as part of their teaching program.

The lessons are written to provide teachers with real-life situations through which students are encouraged to explore mathematical concepts. The editors have developed the original lesson ideas further and used them to exemplify the appropriate use of the technology to explore the mathematical concepts and consolidate their understanding. All keystrokes and screen dumps included in this book are based on the Casio CFX-9850GB Plus graphics calculator. The publication is also available free of charge on the Australian Casio Education Site (ACES), http://www.casio.edu.shriro.com.au.

We are grateful to the following teachers who have submitted the lesson ideas for the editors to develop.

Andrew Armstrong	Laura Gregory	Robert Plant
Kidwell Bolger	Sarah Hamper	Bill Purcell
Maryanna Cox	Logan Hurford	Mike Ristovsky
Geoff Dix	Brenda Komaromi	Lillian Simeoni
Vince Geiger	Greg Ladner	Ian Thomson
Merrilyn Goos	Ramasary Lutchman	Deb Woodard-Knight

The Casio Education Division would also like to thank the editors, Anthony Harradine, Barry Kissane and Gary O'Brien for their efforts and expertise in developing this resource.

We hope that you find this a useful resource, and are pleased to support your professional work in this way. If you have a lesson idea we encourage you to share this with your colleagues. Please contact:

The Casio Education Division
23-27 Chaplin Drive, Lane Cove NSW 2066, Australia
Telephone: 02 9415 5052
Facsimile: 02 9415 5057
Email: casio.edusupport@shriro.com.au
Internet: http://www.casio.edu.shriro.com.au

Guess my rule

Level
Lower secondary

Mathematical ideas
Linear functions, plotting points on a cartesian plane, interpreting graphs

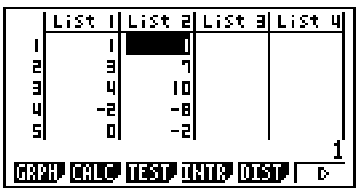
Description and rationale
Many mathematics teachers use the well-known *guess my rule* activity to develop the early ideas of algebra. With the use of a graphics calculator this activity can be enhanced by the use of scatter plots, lists and tables in the construction and checking of the rules.

In the *guess my rule* activity students are shown a machine that accepts some input, completes an operation (or string of operations) on the input and produces an output. Students are normally asked to provide the inputs, with the teacher providing the output. By looking at the input and output students are encouraged to *guess the rule* (or the operations that have occurred on the input). The use of a graphics calculator allows this information to be represented in both pictorial and numerical form. This allows for different ways of thinking to be employed, dependent on the students' preferred mode of learning. There is also the potential to use this activity as an introduction to linear functions. The link between the numerical data and the graphical representation is easily developed and reinforced with the help of the graphics calculator.

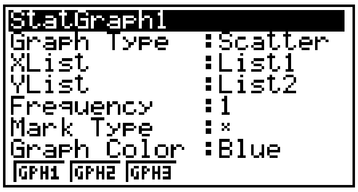
A sample of the data that may be produced by this activity is shown in the table below.

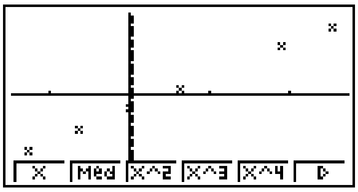
input	output
1	1
3	7
4	10
-2	-8
0	-2
-1	-5

The data can be stored in the graphics calculator lists found in either the STAT mode or the LIST mode. List 1 and List 2 have been used in this case.



In the STAT mode a scatter plot of this data can be produced. This can be done using GRPH (F1) and then SET (F6) to instruct the calculator to draw a scatter plot of List 2 by List 1.

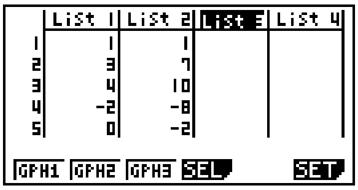




Once a student has determined what they think the rule may be there are a number of methods for checking whether or not their rule is correct.

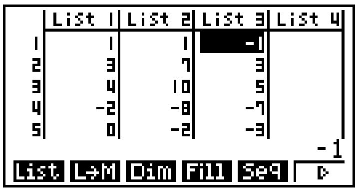
One approach involves performing operations on a list of numbers stored.

To complete an operation on a list of numbers the list name, into which you want the output to go, must first be highlighted; in this case List 3 has been chosen.



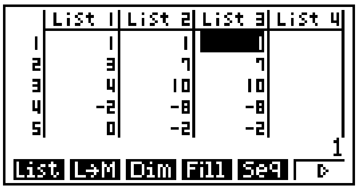
If a student believes the rule is $2x$ (input number) – 3 then the values of the output from this rule can be generated by multiplying the values in List 1 by 2 and then subtracting 3.

To achieve this, ensure the cursor is on the title List 3 and then press OPTN, followed by LIST (F1) to reveal the menu containing the LIST commands. Entering 2 × List 1 - 3 and then pressing EXE will produce the outputs shown.



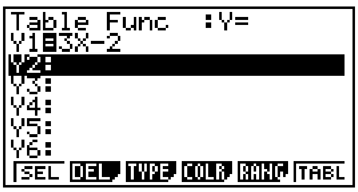
As can be seen by comparing List 2 and List 3 this rule is not correct, a student having made this incorrect guess would then be asked to reassess their rule and repeat the process. Comments about the order and spacing of the inputs may lead to some useful ideas in terms of the best way to approach obtaining the correct rule.

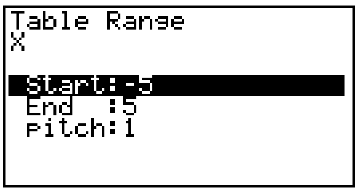
The correct rule for our example is three times the input number subtract two, or 3(List 1) - 2. You can see the output (opposite) that results when the correct rule is entered. List 3 matches List 2 for each of the input numbers in List 1.

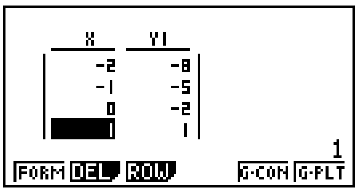


An alternative approach to such a problem may involve students providing the input numbers in an ordered sequence. This slightly more organised approach to the problem may result in students developing a simple technique for obtaining the rule rather than just guessing.

The TABLE mode of the graphics calculator gives rise to a second way to proceed. The rule determined by the student can be entered into the calculator. The *input* is replaced with the pronumeral X. The range for x can be set to include the inputs used by the students and the outputs can then be compared.

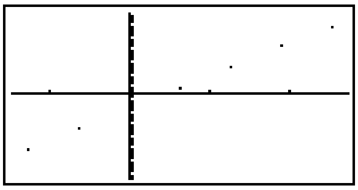


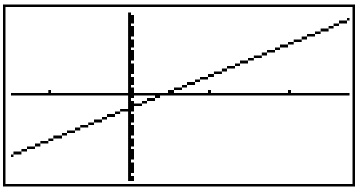




There are two plotting options available in the TABLE mode. The first, plots the values in the table as points, the second connects the points. This is an opportunity to introduce the concepts of continuous and discrete variables.

Two plots produced by these options are shown below.

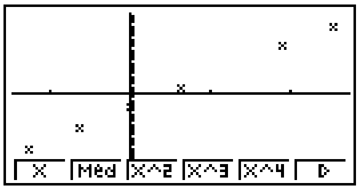




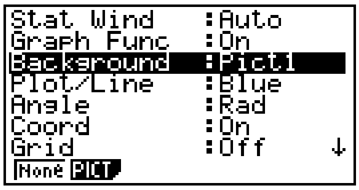
A third approach to determining the rule could involve placing a graph of the rule developed by the student over the top of a scatter plot of output by input values. A correct rule would result in the line covering the points, while an incorrect rule would produce a result where the points are not covered.

This method involves taking a *picture* of the scatter plot and making it the background in the GRAPH mode. The rules developed by the students are then graphed and compared to the original data.

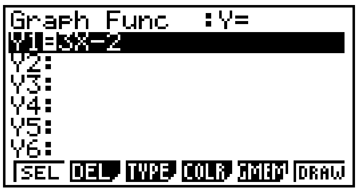
Generate the scatter plot so as to be ready to take a picture of it. With the scatter plot showing, press OPTN and follow the commands on the screen to store the picture (use Pic1).

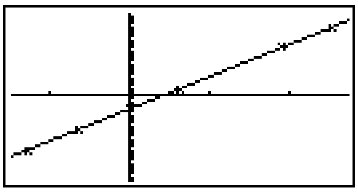


This is placed in the background by going into the SET UP (SHIFT MENU) and altering the background to Pic1 as shown.



A student's rule can then be defined as Y1 and plotted as shown. Make sure that the view window remains unchanged from when the picture was taken as the background is not sensitive to changes in the view window.





Students could be asked to compare and contrast and discuss the limitations of each method outlined above.

Crossing the River

Level

Upper primary and lower secondary

Mathematical ideas

Problem solving, patterning, multiple representation, the need for restricting the domain in certain contexts

Description and rationale

This lesson is derived from the classic puzzle by Henry Ernest Dudeney. The puzzle reads:

> *During the Turkish stampede in Thrace, a small detachment found itself confronted by a wide and deep river that could only be crossed safely in a boat. However, they discovered a boat in which two children were rowing about. It was so small that it would carry only two children or one adult. The detachment consisted of 358 adults. How did the 358 adults cross the river and leave the two children in joint possession of the boat? How many trips (from bank to bank) were made if the least number of trips occurred?*

This puzzle has been changed a little to present to students. The task for the lesson is:

> *A number of adults and children stand on one side of a river that can only be crossed in a boat. They have a small boat that can hold one adult or two children. It is know that 87 trips (bank to bank) are made in order to get the group across the river. It is also know that the least number of trips possible is 87. How many adults and children are in the group?*

This problem offers a wonderful challenge to the students and will develop their abilities in problem solving and patterning. The graphics calculator plays only a minor role – as a medium to display the thinking of the student. However, when using the calculator the student is forced to consider things that traditionally they may not have.

After some thinking the students will be able to find *some* answers to this problem, probably by trial and error. It is likely students will get different answers, leading to the questions

- how many answers are there?
- is there a pattern to the answers?
- is there a simply way to write down all answers?

Student's that approach this problem with a mathematical strategy may have the answer to the above questions as a result of their work, trial and error technicians may not.

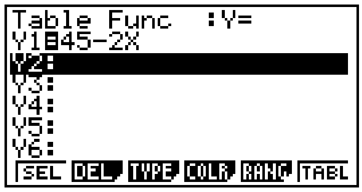
With some more effort students should be able to conjecture that if c is defined as the number of children and a as the number of adults, then

$$c = 45 - 2a$$

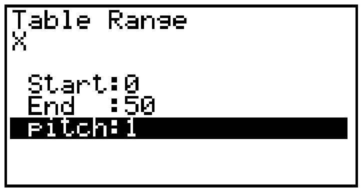
It is left to the reader to verify the truth of this conjecture.
At this point many may not have considered whether this rule is sensible for all values of a.

Once the students have seen the pattern and conjectured its rule they can use the graphics calculator to display the results in a number of ways.

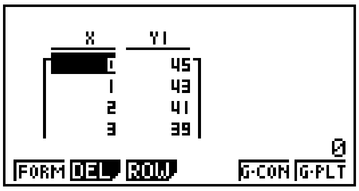
Firstly they could produce a table of values by defining Y1 as the rule they generated.

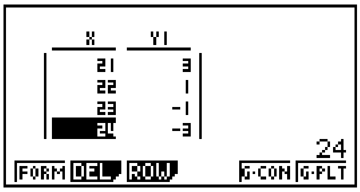


Left to their own devices, the students may well not set the range (domain actually!!) for this rule. If they do, many may do it without thought, as seen opposite. They should be left to explore at this stage.



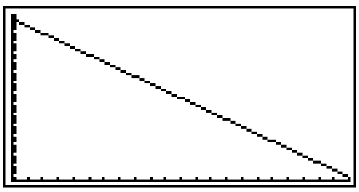
Producing the table and exploring it reveals the following.

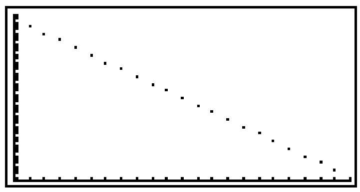




This reopens the question of what is possible and what is not? Most would say 22 adults and one child is fine, but is it? This will help to develop an understanding that when algebraic rules are used to model situations, restrictions may be required so that a sensible model results.

Students can also produce a graphical display of their findings as a nice alternative. Two options are:





The first was produced by G-CON and the other by G-PLT.

Again fruitful discussion can be had as to which of these are appropriate for the situation we have. Most students at an early age seem to believe the line is an appropriate graph in this situation.

This activity can be modified in a number of ways and can also be followed up with more patterning type exercises. You might like to extend the students to cases where the summation of pattern members is required (ie. arithmetic or geometric series). Most students in lower secondary are quite capable of solving senior type problems of this nature if they have the support of a graphics calculator or the like. Hence student's ability to think and form models is developed and they enjoy the feeling of having solved a 'meaty' problem.

How good was Bob Beamon?

Level
Middle and upper secondary

Mathematical ideas
Linear functions, informal data analysis, gradient

Description and Rationale
Mathematical applications and problems found in school textbooks sometimes lack realism and may reinforce the message that mathematics is divorced from real world concerns. Graphics calculators make it easier for students to represent, analyse and interpret data from real life situations, for example through the use of statistical and graphing procedures. This lesson allows secondary students to apply their knowledge of linear functions in the gradient-intercept form $y = mx + c$ to investigate published data on long jump world records. Bob Beamon's astonishing performance in the 1968 Mexico City Olympics provides the context for the task. The calculator is used to draw scatter plots of distance jumped versus the year in which the world record was set. Students then estimate and plot a line of good fit through the data in order to answer the question: *How good was Bob Beamon?*

This lesson also demonstrates how existing resource materials can be given a new lease of life by introducing technology as a means of investigating problems. The task is based on material published by the Spode Group in 1982, before the advent of graphing calculators and the Internet, but makes use of both these technologies to simplify the graphing procedure and place the initial data gathering in the hands of the students themselves.

Background
The task can be introduced by showing an overhead transparency of Beamon in the act of breaking the long jump record (see above), and asking students if they know the identity of the athlete. When trailing this activity with pre-service and practising teachers common responses include Jesse Owens and Carl Lewis, although Beamon is usually recognised as soon as the Mexico City Olympic Games are mentioned. Students can be introduced to the story of Bob Beamon through the newspaper article reproduced on the following page, by kind permission of the *St Petersburg Times*.
(Source: http://www.sptimes.com/News/121699/Sports/Beamon_jumps_into_rec.shtml)

Task
Many studies were published searching for explanations for Beamon's amazing jump, one popular theory attributing his performance to the altitude and rarified air of Mexico City (2250 metres above sea level). It could be said that Beamon was a man ahead of his time ... but how far ahead? In other words – *How good was Bob Beamon?*
Students can analyse long jump world record trends over the course of the 20th century to investigate this question.

Beamon jumps into record book

By BRUCE LOWITT

© *St. Petersburg Times*, published December 16, 1999.

As with most events in track and field, the long jump is measured -- and records are broken -- by fractions of an inch, by centimeters, which is what made Bob Beamon's feat (and, yes, feet) at the 1968 Mexico City Olympics that much more remarkable.

Beamon was, in effect, the No. 2 long-jumper on the U.S. team. Ralph Boston had won gold at the 1960 Rome Olympics and silver at the 1964 Games at Tokyo, and set or tied the world record five times. The record of 27 feet, 4 3/4 inches was Boston's when the 1968 Summer Games started.

Beamon had won 22 of 23 meets he had entered that year, but he was prone to fouling and was considered inconsistent. Furthermore, he had been suspended that June by the Texas-El Paso track team for protesting Brigham Young's Mormon racial policies by refusing to compete against BYU. That left him without a coach.

Boston, who had become Beamon's unofficial coach, and Soviet competitor Igor Ter-Ovanesyan were the favorites in Mexico City. Beamon and Boston were Olympic adversaries, but they were friends and teammates, too.

On Oct. 18, Beamon fouled on his first two qualifying attempts. One more and he would be eliminated.

Boston had a suggestion, something similar to what had happened at the 1936 Berlin Olympics when Jesse Owens had fouled in his first two qualifying attempts and Germany's Luz Long had told him how to avoid another foul. "Ralph Boston did the same for me," Beamon said later. "He told me, 'Bob, you won't foul if you take off a foot behind the foul line. You can't miss.' Basically, that's what Luz Long told Jesse (the German placed a towel at the spot for Owens to use as a takeoff marker), and I took Ralph's advice. I qualified."

With his opening jump in the final, Beamon effectively ended the competition for the gold medal. The 6-foot-3, 160-pound New Yorker sprinted down the runway and launched himself into Mexico City's thin air. When he came down, he was nearly out of the long-jump pit.

"I knew I made a great jump. ... I knew it was more than 27-4 3/4, which was the world record," Beamon said. He ran around excitedly, then fell to his knees, buried his face in his hands and, overcome by the moment, wept.

"I heard some of the guys saying things like 8.9 meters ... or something," he said. "Outside the United States, everything is in meters, so I wasn't sure how far I had jumped. "Then Ralph Boston came over and said, 'Bob, I think it's over 29 feet,' which was almost 2 feet farther than the world record."

From 1961-65, Boston and Ter-Ovanesyan had traded the world record back and forth, raising it from 27-1/2 to 27-2 to 27-3 1/2 to 27-4 3/4. Four and one-quarter inches in nearly five years. "I said to Ralph, 'What happened to 28 feet?'" Beamon said.

After what seemed like an eternity, the public-address announcer made it official: "Bob Beamon's leap, 8.90 meters -- 29 feet, 2 1/2 inches." In one leap he had raised the record by 21 3/4 inches.

"Compared to this jump, we are as children," Ter-Ovanesyan said afterward. And an angry English jumper Lynn Davies said to Beamon, "You have destroyed this event!"

The record would become track and field's oldest, standing for 23 years until Mike Powell leaped 2 inches farther at the 1991 World Championships in Tokyo.

"I've always been a very realistic person," Beamon said the day Powell eclipsed his mark. "I knew the day I set it that eventually someone would come along and surpass it. ... I knew it was inevitable. Now that it has finally happened, I don't feel any different. I don't feel as though something's been taken away from me or that people will think any less of me. "And don't forget," Beamon said with a hint of humor, "I still hold the Olympic record."

-- Information from Times files and 100 Greatest Moments in Olympic History by Bud Greenspan (General Publishing Group) was used in this report.

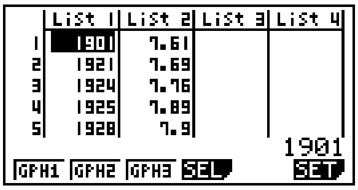
Possible solutions

Students can find data on men's long jump world records from the Internet. A good source is at the URL: http://www.sci.fi/~mapyy/tilastot.html. Here are the official data over the twentieth century:

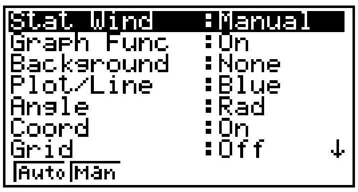
Distance (m)	Name	Country	Date	Place
7.61	Peter O'Connor	GBR	5.8.1901	Dublin
7.69	Edwin Gourdin	USA	23.7.1921	Cambridge, Mass.
7.76	Robert LeGendre	USA	7.7.1924	Paris
7.89	William de Hart Hubbard	USA	13.6.1925	Chicago
7.90	Edward Hamm	USA	7.7.1928	Cambridge, Mass.
7.93	Sylvio Cator	HAI	9.9.1928	Paris
7.98	Chuhei Nambu	JAP	27.10.1931	Tokyo
8.13	Jesse Owens	USA	25.5.1935	Ann Arbor, Mich.
8.21	Ralph Boston	USA	12.8.1960	Walnut, Calif.
8.24	Ralph Boston	USA	27.5.1961	Modesto, Calif.
8.28	Ralph Boston	USA	16.7.1961	Moscow
8.31	Igor Ter-Ovanesyan	URS	10.6.1962	Yerevan, USSR
8.31	Ralph Boston	USA	15.8.1964	Kingston, Jamaica
8.34	Ralph Boston	USA	12.9.1964	Los Angeles
8.35	Ralph Boston	USA	29.5.1965	Modesto, Calif.
8.35	Igor Ter-Ovanesyan	URS	19.10.1967	Mexico City
8.90	Bob Beamon	USA	18.10.1968	Mexico City
8.95	Mike Powell	USA	30.8.91	Tokyo

For an initial look at trends in the data, students may suggest a scatter plot of distance versus year.

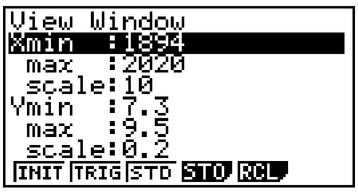
Select the STAT icon from the MENU screen and enter years in List 1 and distances in List 2.



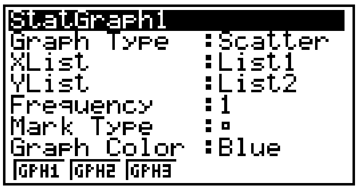
There are benefits in having students choose appropriate settings for the graph viewing window in order to make decisions about domain, range, and scale. Set the Stat Window to Manual (rather than Auto) by selecting SET UP (SHIFT-MENU) then Man (F2).



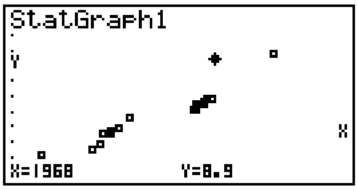
The view window is set via the function buttons below the screen (SHIFT F3) In this case, a good choice for the X-values is shown, with a span of 126 years, suiting the calculator screen dimensions.



The scatter plot is set up via the GRPH (F1) and SET (F6) commands. The set up shown produces a scatter plot of the distance jumped versus the year.

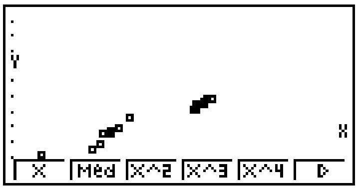


On EXITing the set up and returning to the List screen, selecting GPH1 (F1) produces the scatter plot shown. The TRACE (SHIFT-F1) command can be used to highlight the data point representing Beamon's extraordinary jump.



The scatter plot clearly shows that Beamon's (and Powell's) performance was substantially better than the trend established in previous years. Students may decide to investigate the question of "How good was Beamon?" by predicting the year in which his record jump of 8.90 metres *should* have been expected.

This trend can be more closely examined by deleting the final two data points from Lists 1 and 2, and repeating the procedure for drawing the scatter plot.

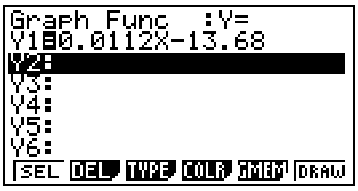


A linear model seems to be suitable for describing the relationship between distance and year. Students should be encouraged to estimate the gradient and find a line of best fit without recourse to the calculator's regression commands, in order to apply their knowledge of linear functions.

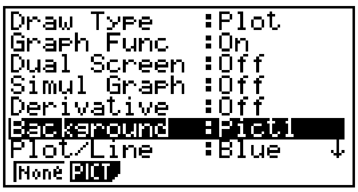
For example, if we select two points (1901, 7.61) and (1967, 8.35), the gradient of the line joining them can be calculated as $m = \frac{y_2 - y_1}{x_2 - x_1} = 0.0112$. Substituting this (rounded) value and the coordinates of one of the points into $y = mx + c$ gives an estimated equation of $y = 0.0112x - 13.68$. This function can then be plotted over the data points, and its appropriateness determined.

With the scatterplot displayed a picture is taken by pressing OPTN and then following the screen commands to store the picture

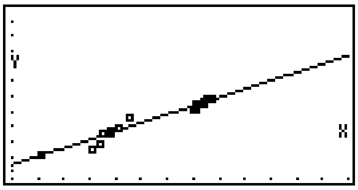
Go back to the MENU and select the GRAPH icon. Type in the equation of the estimated line of good fit, pressing EXE to store it.



The picture is placed in the background by entering the SET UP (SHIFT MENU) and changing the background to the stored picture (here Pict 1).

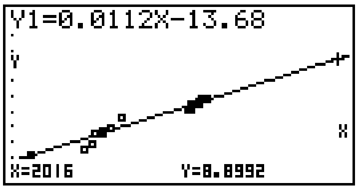


After returning to the GRAPH mode the resulting plot is produced by selecting DRAW (F6).

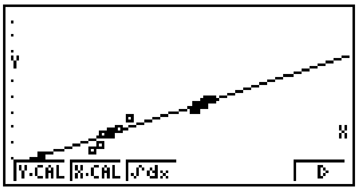


Students may wish to fine tune their equation until they are satisfied with the goodness of fit – this provides opportunities for them to explain how gradient and intercept alterations affect the graph.

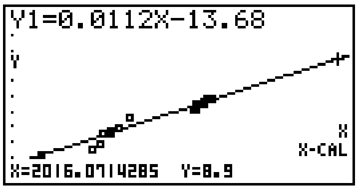
Tracing along the line (SHIFT-F1) allows us to find the year in which a jump of approximately 8.90 metres is predicted, based on the data before Beamon's incredible jump.



A more accurate answer can be obtained by using the G-Solve command (SHIFT-F5). Choose F6 to display the second screen of analysis options, and select X-CAL (F2) since we wish to determine the *x*-coordinate (year) corresponding to a known *y*-coordinate (distance).



Type in 8.90 and press EXE. The cursor moves along the graph until the desired point is reached, and the calculated coordinates are displayed at the bottom of the screen.



This analysis suggests that Beamon was 48 years ahead of his time – the time difference between 1968, when he broke the world record, and 2016, the year in which previous world record trends indicate such a jump would be expected.

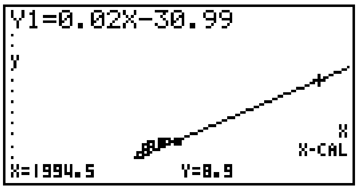
Another way of investigating this situation is by analysing the meaning of the gradient: our function suggests that, on average, the world record increased by 1.12 cm per year from 1901 until 1967. As Beamon increased the existing mark by 55 cm, we could perhaps say that he was around 54 cm ahead of his time!

Some students may argue that a single linear model is not adequate for describing the full set of data. For example, there seem to be two distinct time periods in which the record distance advanced at different rates: the first a time of rapid improvement from 1921 to 1935, and the second a much more gradual change from 1960 until 1967. It would be interesting to speculate on the reasons for this difference, and why Jesse Owens's record stood unchallenged for so long. (The Second World War obviously had an effect on international sporting competition; for example, the Olympic Games were not held between 1936 and 1948.)

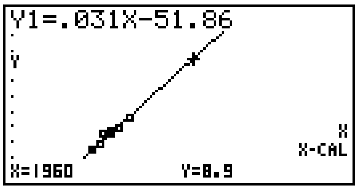
The calculator procedures outlined above can be repeated to separately analyse the data in these two time periods. For example, use Lists 3 and 4 for 1921-1935 records, and Lists 5 and 6 for 1960-1967 records. Before viewing scatter plots, change the GRAPH background back to to None. Store StatGraph2 (Lists 3 and 4) as Pic2 and StatGraph3

(Lists 5 and 6) as Pic3. Re-set the GRAPH background to Pic2 or Pic3 when plotting estimated lines of good fit over the scatter plots.

If we look first at the 1960-1967 data, a line of good fit can be calculated as $y = 0.02x - 30.99$. With the world record distance increasing as an average rate of 20 cm per year during this period, the equivalent of Bob Beamon's jump of 8.90 metres is predicted in the year 1994 – so was he only 26 years ahead of his time?



However, the data for 1921-1935 are even more interesting, as during this time frame the distance was advancing at 31 cm per year. Our line of good fit, $y = 0.031x - 51.86$, predicts a leap of 8.90 metres in 1960! What does this say about the danger of extrapolation far beyond the data provided?



Extensions

Sporting world records lend themselves well to this type of analysis, and other similar tasks could be easily devised – perhaps by the students themselves, using the Internet site noted above as a source of data. A new world record provides an ideal opportunity to assess the adequacy of a model developed from previous data – that is, how well did the model predict the new record?

Students could also consider whether a linear model is likely to continue to be the most appropriate one for describing human performance. Will a plateau be reached, or will improvements in training regimes, sports science, nutrition and so one lead to corresponding improvement in sporting achievement? The issue of drug use in sport arises naturally from such questions, demonstrating the potential for mathematics to be used in analysing social problems. However, Bob Beamon's feat should remind us that mathematical models are at best approximations requiring intelligent interpretation as much as mathematical knowledge and skill.

Explorations of these kinds have considerable potential to help students understand the nature of mathematical modelling. The graphics calculator is an invaluable tool to support such work, allowing students to take charge of the modelling process.

Reference

Spode Group (1982). *Solving real problems with mathematics. Vol. 2.* Cranfield, England: CIT Press.

Learning about quadratic equations through programming

Level

Middle to upper secondary

Mathematical Ideas

Solving quadratic equations, the discriminant, the quadratic formula, programming, problem solving

Description/Rationale

There are many purposes a teacher may have for using a graphic calculator. This activity focuses on using the calculator to perform menial or repetitive tasks, to consolidate and extend knowledge and to assist in inferring results.

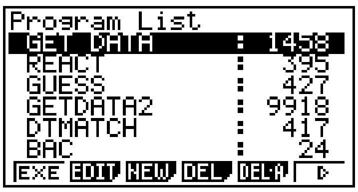
Although EQUA mode of the Casio series of calculators provides numeric solutions to quadratic equations, students cannot see how the calculator produces these. A short program, such as one described here, will make the details of the procedure more transparent, as well as providing important information about the discriminant that is not provided explicitly by the inbuilt routine.

Programming the quadratic formula

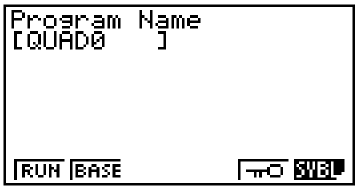
Computers can be programmed to perform menial and repetitive tasks. To do so though, a formula or algorithm is needed; one that works in all cases (or in most cases and we need to know the cases for which it does not). The quadratic formula is one common example of such a formula.

Having a program in the calculator that will compute the roots of any quadratic equation of the form $ax^2 + bx + c = 0$ not only allows students to perform a menial task with ease but opens up a world of activities that many may not have considered. It also allows us to move on to some exciting mathematics, as you will see later in this book.

Enter the PRGM mode of the calculator (which stands for program). If you already have some programs in the calculator the window will look as seen opposite. If not it will be blank.



Use NEW (F3) and then enter a name for the program, the flashing A tells you that the alpha lock is on, so just type the name using the red letters above the keys as a guide. I used QUAD0.
Press EXE and you are ready to enter the code.



The program code is as follows. The text to the right explains the code.

`"A"?→A↵`	[Requests the leading coefficient of the equation]
`"B"?→B↵`	[Requests the leading coefficient of the equation]
`"C"?→C↵`	[Requests the leading coefficient of the equation]
`B^2−4AC→D↵`	[Calculates the discriminant and sets D at this value]
`"DISCRIMINANT":D◢`	[Prints the word discriminant and the value]
`(−B+√D)÷2A→E↵`	[calculates one root and sets E at this value]
`(−B−√D)÷2A→F↵`	[calculates the other root and sets F at this value]
`"ROOTS ARE"↵`	[displays ROOTS on the screen]
`E◢`	[displays one of the roots on the screen]
`"AND"↵`	[displays AND on the screen]
`F↵`	[displays the other root on the screen]

To enter code:
press SHIFT then ALPHA to turn the alpha lock on. Note the symbols at the base of the screen. Enter " (F2) and then type A and enter another " (F2). Then use PRGM (SHIFT then VARS) to access the ? symbol. Enter this symbol (F4) and then press the → key above the AC/ON key and then enter A (ALPHA then X,θ,T key) and then press EXE to complete the first line of code.

Note that after pressing EXE a bent arrow (↵) appears at the end of the line to denote the line of code has ended.

Now we need to access the quotation symbol again. Press EXIT to return to the home screen for the PRGM module. Use SYBL (F6) to access the quotation symbol (last time we just used Alpha Lock, either works). Now enter the first three lines of code.

You should now be able to complete the rest of the code. However, note that the ◢ symbol (the output command) is used at the end of the lines where we want the calculator to tell us the result of a calculation before continuing. It can be accessed in a similar way to the ? symbol (SHIFT then VARS then F5). The : symbol is also accessed in this way, but you will need to use the continuation key (F6) to reveal it. Complete the code entry.

When you are finished, press EXIT twice to return to the screen seen opposite and, with the name of the program highlighted, press EXE to run the program. When A? is displayed the program is asking you to input the leading coefficient of the quadratic. Type the value, press EXE and then proceed.

When – Disp – is displayed on the screen simply press EXE to continue.

A sample output is:

```
4
DISCRIMINANT
                0
ROOTS ARE
               -2
AND
               -2
```

There are many types of tasks that students could be asked to do if they have such a program in their calculator. The tasks range from the simplest of root determinations to rather complex problems.

Consider each of the following tasks. Comments follow each of the tasks.

Task One
Use your program to solve the following quadratic equations:

a) $2x^2 + 8x - 4 = 0$

b) $5x^2 + 10x - 10 = 0$

c) $-5x^2 + 15x = 5$

d) $-2x^2 + 5x = -7$

e) $\frac{1}{2}x^2 = 6x + 5$

f) $\frac{1}{x} + 2 = 5x$

This task is somewhat simple but still requires that students perform some algebra before using the program. It reinforces the notion of appropriate form.

Task Two

a) Using your program, solve four quadratic equations that have a discriminant of 0. What do you notice? Explain your observation.

b) Using your program, solve four quadratic equations that have a discriminant that is a perfect square. What do you notice? Explain your observation.

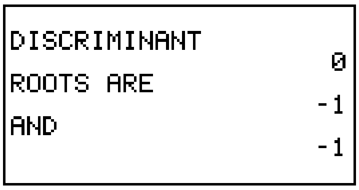
c) Using your program, solve four quadratic equations that have a discriminant that is positive but *not* a perfect square. What do you notice? Explain your observation.

d) Using your program, solve four quadratic equations that have a discriminant that is negative. What do you notice? Explain your observation.

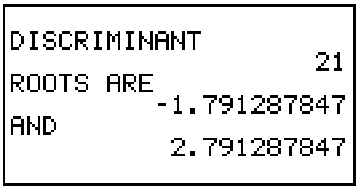
This task would be best given to students who had not yet investigated the discriminant in any great depth. It requires students to construct their own quadratic equations given certain conditions and then to observe the results carefully to notice the neat outcomes. It offers a fresh way to reinforce the use of the discriminant and the knowledge that goes with its use.

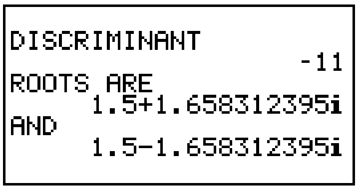
Task Three

For each of the sample outputs below, determine the equation whose coefficients were entered into the calculator.

a) 

b) 

c) 

d) 

This task will reinforce the idea that an infinite number of quadratics have the same zeros but supplies the added challenge of investigating whether or not knowing the discriminant narrows the field. Part c) and d) will give students something to really think about. Students could make up their own screens and compete against their partner in determining the correct quadratic equation(s).

Families of polar curves

Level

Upper secondary

Mathematical Ideas

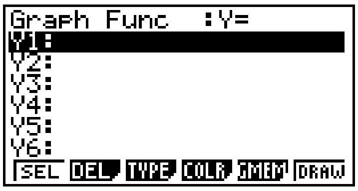
Polar coordinates, families of polar curves, properties of trigonometric functions

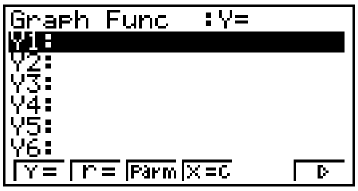
Description and Rationale

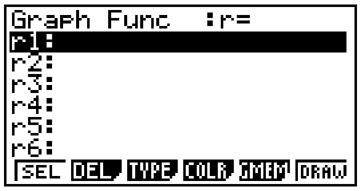
A graphics calculator allows for the efficient investigation of families of curves. One such family is the polar curves. Polar curves are based on the polar coordinate system defined by (r, θ), a distance from the origin (r) in a particular direction (θ) anticlockwise from the positive x-axis. Exploring and attempting to explain some of these patterns provides many opportunities for students to enhance and communicate their mathematical understanding. The stimulation generated by the opportunity to create very interesting patterns acts as a great source of motivation for students.

A sample of what can be done follows.

After entering the GRAPH mode the graph TYPE (F3) needs to be altered to polar coordinates. Press r = (F2) to enable the rules for polar curves to be described. The calculator needs to be in radians rather than degrees. This can be checked in the SET UP (SHIFT MENU) and changed if required.

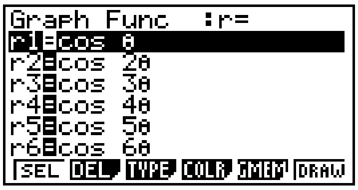




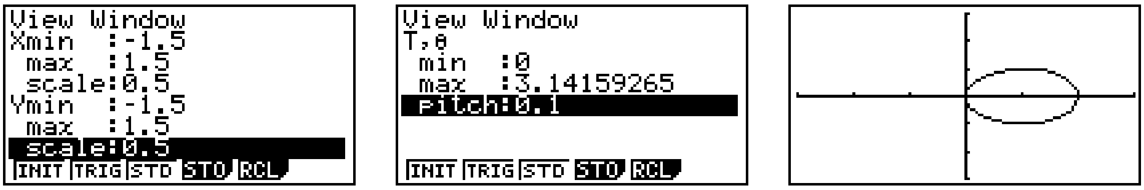


The family defined by $r = \cos n\theta$ will be investigated.

The first 6 members of this family have been defined and will be graphed one at a time. With the calculator in polar mode the θ is generated by the X, θ ,T key.



Before graphing, an appropriate view window must be selected. It can be quite challenging for students to select a good view window. They should, however, be encouraged to determine the appropriate view window rather than simply guess and check. It is also important for the students to consider the appropriate values to define for θ. This is not a concern when dealing with the plotting of rectangular coordinates. Students will need to consider what is an appropriate setting (both in terms of the minimum, maximum and pitch for θ) and how the X min, X max and so on relate to the settings for θ. In our example, should the view window be set as shown below, the graph of $r = \cos\theta$ that results will look, to the knowing, a little distorted. The uninitiated may accept it should look oval in shape. This would be most unfortunate.

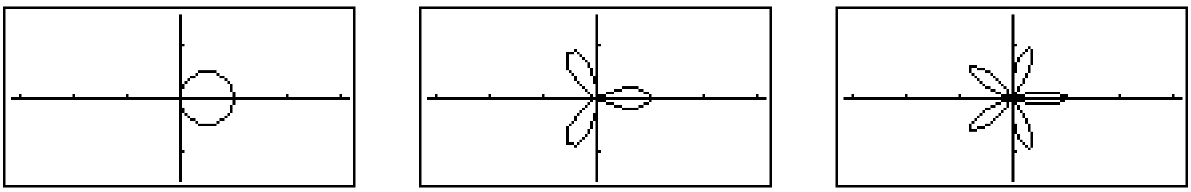


If a slight change is made to the settings for θ, some other interesting outcomes result for the graph of $r = \cos\theta$.

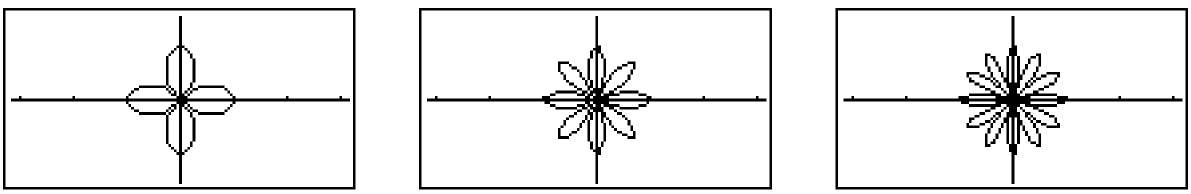
It is critical that, when interested in the geometry of the graph a graphics calculator produces, we set a view window that we could call *square*. That is, the same distance that represents 1 unit on the vertical axis represents 1 unit on the horizontal axis. To achieve this we can select the INIT (initial) view window setting when we select the view window command. After drawing the graph we ZOOM (F2) IN or OUT and the window will remain square so long as the vertical and horizontal zoom factors are the same (accessed by selecting FACT (F2)).

Graphs of the family defined earlier (using an appropriate view window) are shown below. They are categorised by whether n is odd or even. Note that the congruency of the petals within one graph could be conjectured so long as a *square* view window is chosen.

n odd

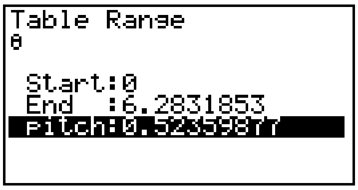


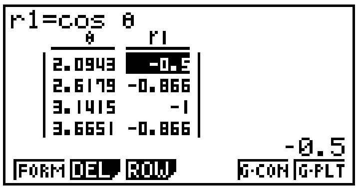
n even



Students could reasonably conjecture that for those where n is an odd number a pattern with n petals is produced and for those where n is an even number a pattern with $2n$ petals is produced.

Students should be encouraged to attempt to explain why these particular patterns are produced. Such activities lead to a deeper and better understanding of the mathematical concepts involved. To help in this quest, students could be directed to consider the actual values being plotted on these graphs. By entering the TABLE mode and setting the RANG (F5) for θ as shown, students will be able to view the values generated and explore and reinforce their ideas. The pitch has been set at $\pi/6$ radians and the end value is 2π radians.





A similar investigation for the family with form $r = \sin n\theta$ could follow. Attempts to compare the graphs and values of these two families would allow students to discover links between the cosine and sine functions.

There are many families of polar curves that are worthy of investigation. Examples include:

$$r = a - b\cos\theta$$
$$r = a - a\cos\theta$$
$$r = a(\cos 2\theta)^{\frac{1}{2}}$$

Creating histograms

Level
Middle or upper secondary

Mathematical Ideas
Students will be able to to display histograms as a prelude to simple data analysis. Class intervals, mid-points, frequency polygons, histograms, relative frequencies and percentage frequencies are all involved.

Description and rationale
A histogram is very useful way of displaying single variable data but they can be time consuming to draw manually. A calculator can aid greatly here as you can use it to quickly create histograms for any data you need to analyse. In some situations, especially when students have not collected their own data, data are already grouped into class intervals. The lesson works through an example involving grouped data (for motorcycle deaths for five year age cohorts) showing some of the different ways histograms can be displayed on the calculator and how to produce these. The lesson deliberately creates some of the problems that students will come across, notably the calculator's choice of incorrect intervals.

Aspects covered include histograms, frequency polygons, relative frequency histograms and percentage histograms. It also shows students how to use the LIST commands in the option menu to calculate relative frequencies and percentage frequencies automatically and place them in a new list. This is a valuable skill generally, allowing for other kinds of data transformations to be carried out.

The lesson was designed for use with a popular Western Australian Year 11 text (*Foundations of Maths* by Alan Sadler), from which the data were obtained, in order to show students at the start of a data analysis chapter how to use the calculator for histograms. The lesson worked well as a prelude to the students then working through the questions in this section of the text. The instructions that follow have been written for students to use directly. They assume that students already have some familiarity with basic calculator functions, such as making choice from menus, but are unfamiliar with the construction of histograms in particular.

Reference
Sadler, Alan (2000) *Foundations of Mathematics*, Sadler Family Trust.

How to create a histogram from grouped data

Histograms offer an important way to represent univariate data. A calculator is a powerful tool to create histograms, either using your own data or those from someone else (such as a textbook). A good way to develop the necessary calculator skill is to work through an example carefully, as we do here. The data in this case are in the textbook *Foundations of Mathematics*, by Alan Sadler (published by the Alan Sadler Family Trust).

The road accident statistics for a country for one year showed that 186 motorcyclists (drivers not passengers) in the age range fifteen to fifty-nine, had died in road accidents. The distribution of the ages of these riders is shown in columns 2 & 3 of the table below.

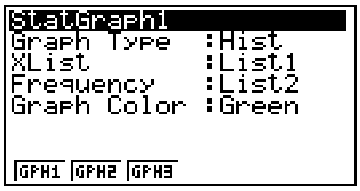
Column 1	Column 2	Column 3	Column 4	Column 5
Midpoints	Age (x years)	Drivers Killed	Relative Frequency	Percentage Frequency
17.5	$15 \le x < 20$	40	0.215	21.5%
22.5	$20 \le x < 25$	59		
27.5	$25 \le x < 30$	29		
32.5	$30 \le x < 35$	19		
	$35 \le x < 40$	16		
	$40 \le x < 45$	11		
	$45 \le x < 50$	8		
	$50 \le x < 55$	2		
	$55 \le x < 60$	2		
	Total	186	1	100

Usually, of course, a first step in drawing histograms is to choose suitable intervals for grouping the data; in this case, there is no choice, as the data have already been grouped. The table displays the data as grouped data or in class intervals (shown in column 2). We cannot enter grouped data directly into the calculator, so we must instead use the *midpoints* of the class intervals. Some of these have been added in column 1 of the table.

Fill in the rest of the midpoints on the table in column 1 and enter them into List 1 and Drivers Killed (column 3) into List 2 via the STAT mode (clear any previous data in List 1 and List 2 (press F6 then use DEL A (F4) for each list).

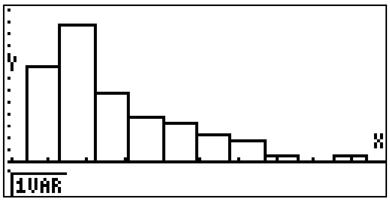
Now press GRPH (F1) and SET (F6) in order to set the histogram you wish to draw. Change the display to match the below using the arrow keys and F1 to F6 keys.

Notice that Graph 1 is being defined.
The type of graph is a histogram (F6 then F1).
The X list shows the data, in this case the midpoints.
The frequency for each interval is shown as List 2.
You can choose any colour you like, of course.



Press EXIT when finished.
When drawing a histogram, you can choose your own scales or allow the calculator to do this for you. We will firstly look at the automatic option. Use SET UP (SHIFT and MENU) to select *Auto* instead of *Manual* for the Statistics Window (which is at the top of the menu)
Press EXIT when finished.

Press F1 for GRPH and again F1 for GPH1 to DRAW the graph with the calculator automatically making decisions about the scale and the intervals. Your histogram should look like the one below.



We have a problem!

Look at the original data and the histogram (particularly the last three columns on the right) and see if you can identify the problem.

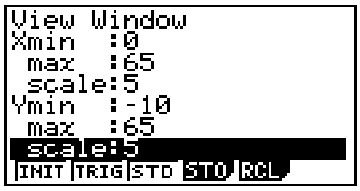
You can TRACE the histogram (with SHIFT and F1), using the arrow keys to move along the bars. Notice the “X=   ” values (bottom left of screen) which show the starting points of the histogram bars (even though they appear in the middle of each column). Are these the same as the interval starting points in the table (ie. 15, 20, 25 etc.)?

You should have noted that the starting points of the bars are *not* the same as the starting points of the intervals in our example (ie.15, 20, 25 etc.). The calculator histogram has an empty column at X = 53 which is not representative of our original data. The reason for this is that the calculator has automatically determined intervals that do not match the data entered.

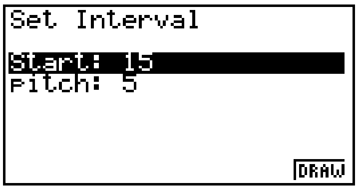
Whenever you create a histogram that has unwanted gaps in it, this is usually the cause.

Setting intervals manually

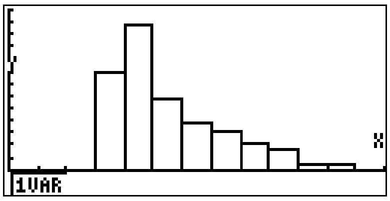
To fix this problem we need to set the histogram’s intervals manually.
Use SET UP to change the Stat Wind display to *Manual* instead of *Auto*. The calculator will now expect you to choose your own scale values for the histogram, using the View Window. (If you do not do this, it will use the values already set; in some cases, this will mean that you do not see a graph at all.)
To see how this works, use the View Window menu (SHIFT and F3) to enter the settings shown below.



Xmin could be set as 14 but 0 will show the *y*-axis.
Xmax is set a bit beyond the end of our largest interval.
Ymin could be zero but then *x*-axis disappears.
Ymax is set a bit beyond our highest frequency.
The scale of 5 on each axis is matter of choice: 1, 2 or 10 would also work
Exit from the View Window menu and press F1 for GRPH and again F1 GPH1 (which we set up previously).
It now asks to Set Interval (because we are in manual Stat Window) so make the choices shown below, which match the original data. The *Start* is the starting point of the first interval and the *pitch* is the width of each interval (in this case, 5).



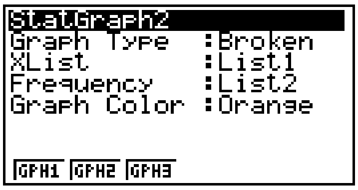
After making these choices, DRAW the histogram (using F6).
The graph should now effectively be the same as the one below.



Note that the graph now has no empty columns and represents the data faithfully. If you now trace the graph as before, you will see they are the appropriate starting points or lower boundaries of our intervals (ie. 15, 20, 25 etc.).

Graphing a frequency polygon for the same data

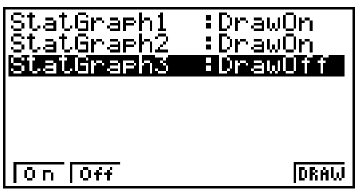
An alternative way of displaying this data would be to use a *frequency polygon*. The screen below shows how to set up the second statistics graph for this purpose. Start with GRAPH and then SET. Make sure you choose GPH2 (using F2) to set the graph. The calculator uses the term 'Broken' instead of 'Frequency Polygon'.
In this case, we have chosen a different colour for the frequency polygon, so that it will stand out from the histogram.



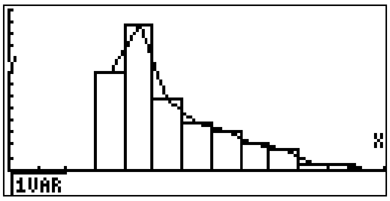
After completing these settings, press EXIT and then draw the frequency polygon, using GPH2. Leave the intervals the same as for the histogram. Notice that the polygon merely joins the tops of the histogram bars together with line segments (and is not actually a polygon, since it starts with the first bar, rather than the *x*-axis).

Drawing the histogram and frequency polygon together

Often histograms and frequency polygons are drawn together on the same set of axes, especially as they are so strongly related.
To do this, choose SEL (F4) instead of a graph to draw and select which graphs you wish to draw at the same time. In this case, the two graphs are StatGraph1 and StatGraph2. The next screen shows that these two have both been turned on (while StatGraph3 has been left turned off.)



Use DRAW to draw both graphs together. Your screen should look like the one below.



Note that the broken line plot does not start and finish on the *x*–axis in the previous empty class interval as frequency polygons often do. This is really a difference in style but something of which to be aware.

Both the histogram ad the frequency polygon show a similar important aspect of these data: that motorcyclist deaths are most prevalent among younger drivers.

Calculating relative frequencies and percentages from frequency data

So far, on the histograms we have drawn, the number of drivers killed has been shown on the y-axis. Sometimes we want to display instead the proportion or percentage (in this case of drivers killed) on the *y*-axis. That is, instead of frequencies, we wish to use *relative* frequencies (or sometimes percentage frequencies)

To work out relative frequencies:
Relative Frequency = Frequency ÷ Total

For example, the frequency for the $15 \le x < 20$ interval is 40.
So the relative frequency is 40 ÷ 186 = 0.215.
(Note that the total, 186, is the sum of all the frequencies.)

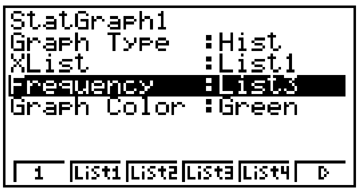
To work out percentages:
Percentage frequency = Relative Frequency × 100

For example, for the $15 \le x < 20$ interval the relative frequency is 0.215.
So the percentage frequency is 0.215 × 100 = 21.5%. That is, 21.5% of the drivers killed were in the age group from 15 up to 20.

Work out the relative frequencies and percentage frequencies now and enter them into columns 4 and 5 of the table on the first page.

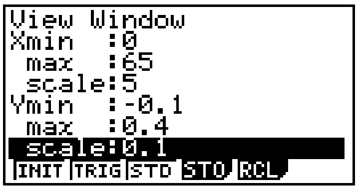
To create a relative frequency histogram

Enter the relative frequencies you have worked out into List 3 of the calculator. Change StatGraph 1 so that it uses these as the frequencies rather than the frequencies previously set. The revised definition is shown below.

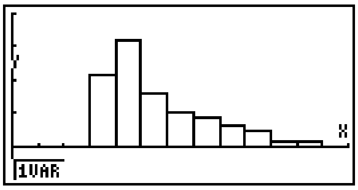


Press EXIT when you are finished.

In Manual mode, you will also need to change the View Window settings (otherwise the histogram will look very small). Use the values shown below; only the Y values need to be changed, to accommodate the new relative frequencies to be represented.



When the graph is drawn, it will look like this (much the same as previously).



You will notice a difference if you TRCE the histogram, as the " f=   " values now show relative frequencies, rather than frequencies.

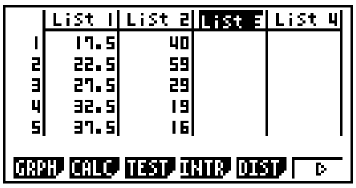
The percentage frequency histogram could be done following a similar procedure, after entering the percentage frequencies into List 4 and re-defining StatGraph1 accordingly. This time change the Y range to about -10 ≤ y < 40, with a scale of 5, for the percentage frequencies. Try this for yourself, and notice that he shape is similar, although the percentage frequencies are shown when you trace.

Calculating relative and percentage frequencies quickly

Calculating relative frequencies and percentage frequencies as you did above is a bit tedious and it is easy to make an error since many calculations are involved. A better strategy is to get the calculator to do this for you.

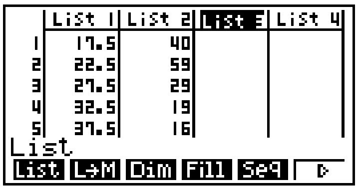
Start by deleting the data previously placed into List 3 and List 4. (It is these that we will calculate automatically.)

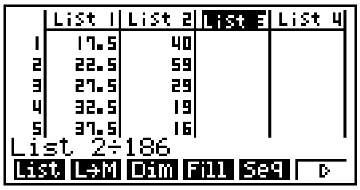
Now highlight an empty list into which the calculated relative frequencies will be stored. To do this, move the cursor to the name of the list at the top of the screen, as shown in the next screen. (Notice that the name of the list is shown in reverse print).



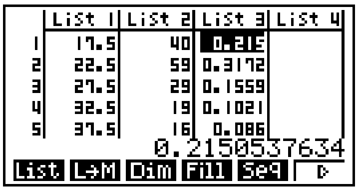
We are going to define List 3 as List 2 divided by 186. That is, the relative frequency for each interval is the frequency divided by 186, as you did by hand. To do this press OPTN then LIST (F1) and List (F1) again.

The word List should appear near the bottom of the screen. Use it to complete the definition as shown on the two screens below.





Press EXE (This divides each value in List 2 by the total 186 and stores it into the list you highlighted, in this case List 3.) The results are shown below.



Compare the values calculated in List 3 with the values you manually calculated earlier and wrote into the table on the first page.

The same sort of process can be used to calculate the percentage frequencies and store them into List 4.

There are two ways of doing this. Calculate *List 2÷186×100* or *List 3×100* , provided you have already calculated List 3 relative frequencies. Try both ways to make sure that you can see how this works.

You should now be equipped to handle most situations requiring histograms quickly and easily on your calculator, and can use your time in *interpreting* the graphs that the calculator produces for you – the most important part of data analysis.

Using a program for integration

Level
Upper secondary

Mathematical ideas
Integration, Riemann sums, limits, programming

Description and Rationale
My Introductory Calculus students were just starting on the topic of Integration. I certainly remember lessons from my own high school years calculating areas of rectangles – only to fully grasp the whole idea days later when the calculations were complete. Naturally, my hope was for my students to 'get the point' without the hindrance of the numerous calculations.

A colleague showed me a program *INTAREA* written by David Tynan and John Dowsey. The program appeared as a Workshop Sample Task in VCE Graphics Calculator Project, and is available at the ACES web site for downloading. This program, after receiving inputs of the function, integral bounds, number of intervals will calculate the over-estimate, under-estimate, sums from using the Midpoint rule and Trapezoidal rule and a final 'best estimate'. With all this done in a matter of minutes, my students could see the purpose of the lesson without needing to complete tedious calculations.

In fact, with each closer approximation, obtained by increasing the number of intervals involved. Involved, they waited eagerly to see the result. I was very impressed with this program when I saw it. Perhaps it was the fact that I could barely contain my own excitement and enthusiasm but what I didn't expect was the reaction from my students. They actually applauded the lesson and I was fortunate enough to hear those words we as teachers often only dream of : "Thanks, Miss, I really enjoyed that lesson". Despite the fact that it was their lunch time, more than half the class queued at my desk for a copy of the program!

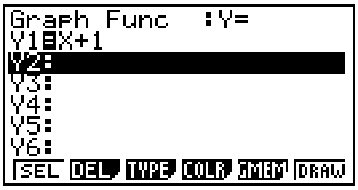
Using the program
There are many ways that this program can be used by either students or their teachers. My first use was in demonstration mode, but students can use it effectively on an individual basis, especially after they have seen it used in class.

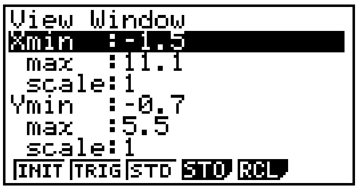
Essentially, you enter a function Y1 into the function list in GRAPH mode and graph it on a suitable set of axes. Then run the program, which asks for the limits of integration and the number of intervals to be chosen to approximate the integral. In succession, the program shows the area under the curve when left and right intervals are chosen, as well as intervals centred on the midpoints.

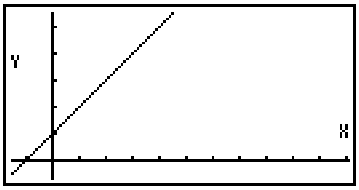
After this, the program uses the trapezoidal rule to approximate the area better and finally produces a 'best' approximation. The idea of approximating the area by rectangles is beautifully illustrated on the screen, and it is clear how the left, right and midpoint approximations are related. By choosing increasingly larger numbers of intervals, the gap between these closes, of course, which is an important understanding for students to acquire. If a large number of intervals is chosen, the area under the curve appears to be almost a solid area in these various procedures, and the variation among the estimates reduces.

To illustrate, here is a simple example, using an elementary linear function (under which students can easily calculate the area of 12 square units by finding the area of the polygonal shape concerned).

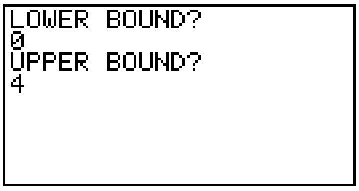
We start by drawing the graph of $y = x + 1$ in the first quadrant:



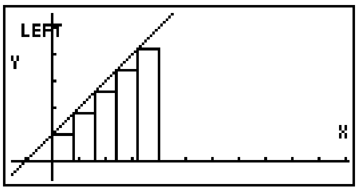


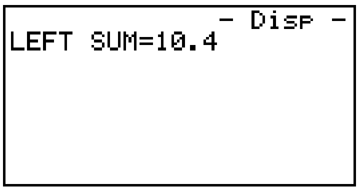


To consider the area under the line from $x = 0$ to $x = 4$, start the program and enter the limits of 0 and 4, pressing EXE after each:



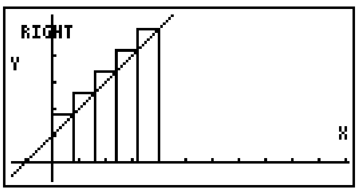
To see the approximation process, start with a small number of intervals. In this case, we chose 5. The calculator starts by drawing five rectangles from the left endpoints, and giving their total area after EXE is pressed:

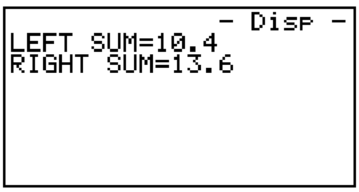




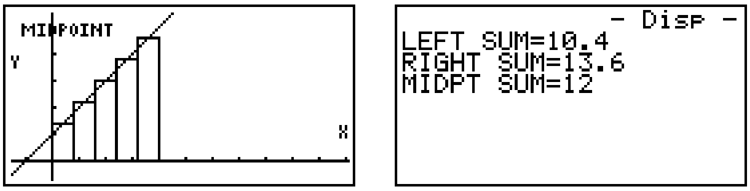
Clearly, the rectangles on the left intervals underestimate the area under the line, which is seen to be larger than 10.4.

Press EXE to continue with the rightmost rectangles on the same five intervals. The sum of 13.6 is now clearly larger than the area sought:



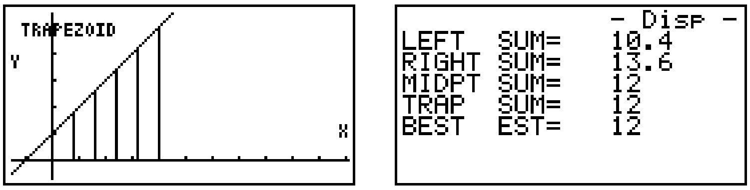


The midpoint of the intervals is provided next, after EXE is pressed.



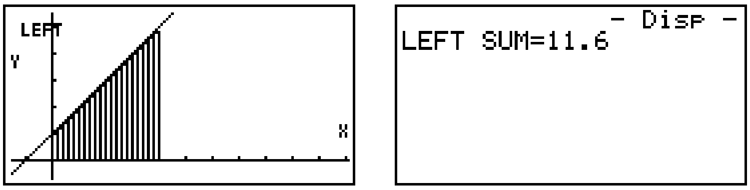
In this case, the area obtained by summing rectangles at the midpoints of the intervals is precisely the average of the left and right sums (since the 'curve' is a line with unit slope), although this is not generally the case.

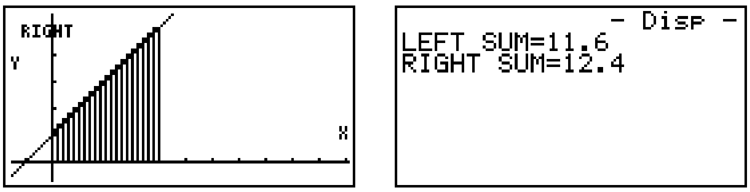
The next case uses the trapezoidal rule, clearly an excellent fit here, as the shape under the curve is actually a trapezium each time:



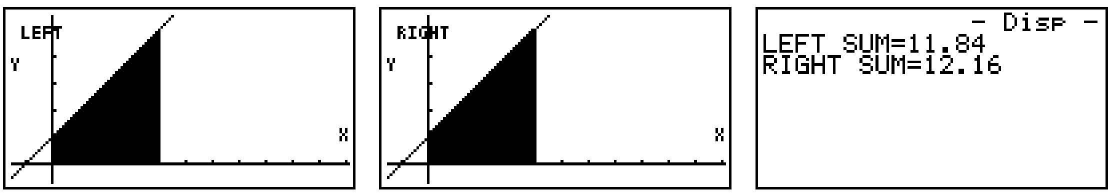
The best estimate in this case is the same as the last two, although again this is not generally so. The essential ideas are beautifully conveyed by this program, which students found easy to use and convincing.

Of course, a larger number of intervals gives much tighter results. Here is what happens with 20 intervals:



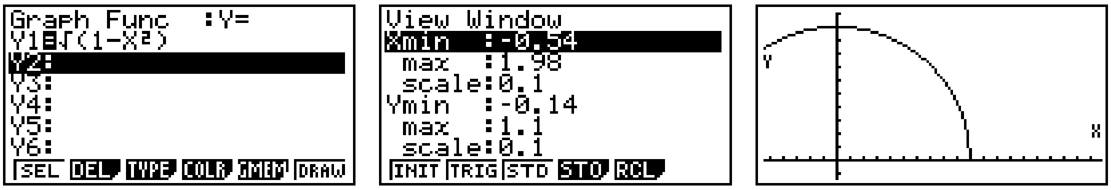


An even larger number of intervals produces even more convincing results, at the expense of taking much more time. The rectangles below are based on 50 intervals, and scarcely distinguishable visually, if at all, although slightly different numerically:

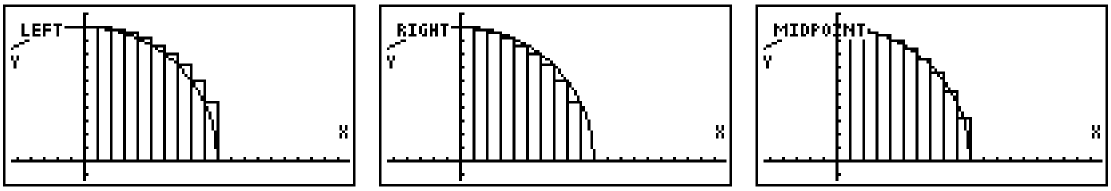


The power of the method is seen when functions are chosen for which geometric alternatives are not available, of course. For example, finding the area under the curve given by y = √(1 - x2) between x = 0 and x = 1 gives an approximation to the area of a quadrant of a circle of radius 1, or π/4.

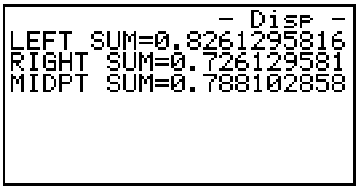
The next screens show what happens for a small number (10) of rectangles. We start by drawing the curve on a sufficiently large scale to show clearly the rectangles chosen:



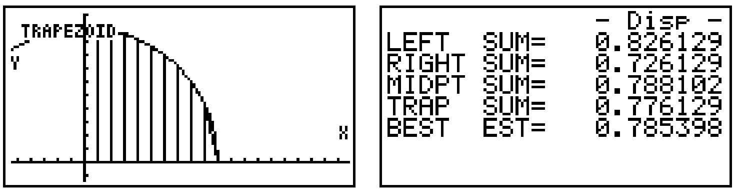
Here is what happens with ten rectangles:



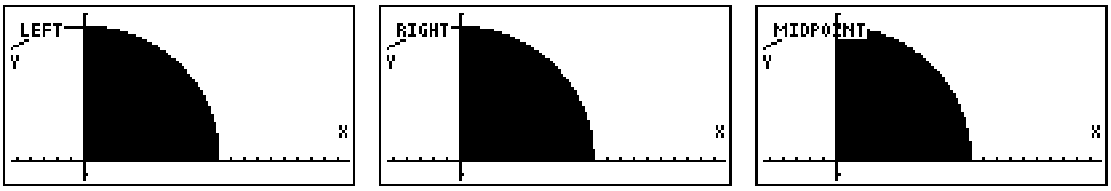
The numerical values are close to each other, with the midpoint close to π/4:



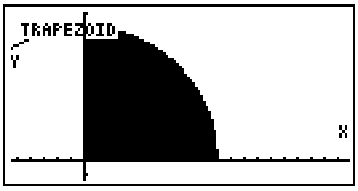
The trapezoidal rule also gives a result close to the theoretical value:

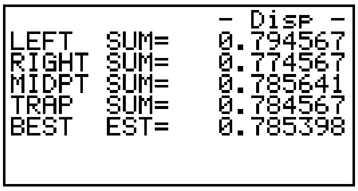


Of course, the approximations become very close with a large number of intervals. the screens below show the case for 50 intervals, for which all the screens look the same:

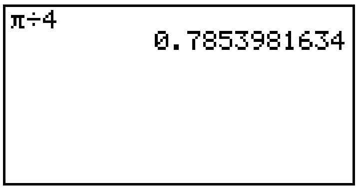


Of course, they are slightly different numerically (which is a good lesson for students to learn … not to rely entirely on the visual element), as the screen on the right below shows:



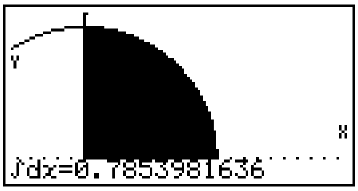


By the time they begin to study the idea of integration as the area under a curve, students are very familiar with the area of a circle, and can readily check on their calculator that the theoretical area of the quadrant is $\pi/4 \approx 0.785398\ldots$

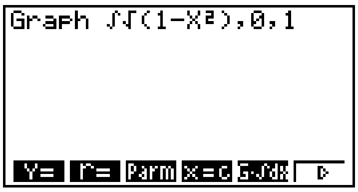


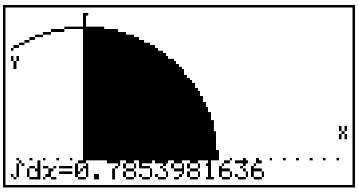
The trapezoidal rule gives an excellent approximation to this and the 'best' method (using the method used by the calculator internally) performs even better.

Later, the students can use the G-SOLVE procedure available in GRAPH mode to obtain the numerical integral, as the next screen shows. With the experience obtained from using the INTAREA program, they are now more likely to understand what the routine is doing and why it takes so long to produce an answer):



An alternative is to use the integral graph command in Sketch mode (Start by pressing Sketch (F4) in RUN mode):





As I have mentioned before, there is a huge wealth of ideas and resources ‘out there’ and this program is only one of many provided on the ACES website. Of course, there needs to be research and planning time (especially the ‘playing around and familiarise with the equipment time!) in any lesson incorporating technology but such resources are certainly valuable in enriching the learning experience for our students and for ourselves.

Reference

Tynan, D. & Dowsey, J. (1997) *INTAREA program*, ACES website. [http://www.casio.edu.shriro.com.au/prog_page.html]

Financial mathematics

Level
Upper secondary

Mathematical ideas
Recursive sequences, annuities, simple interest, compound interest, reducible interest

Description and Rationale
Students in our Year 12 unit (*Discrete Mathematics*) study various aspects of recursion, which are used to inform their study of aspects of financial mathematics. The relevant official syllabus (Curriculum Council of WA, 2001) contains the following entries:

- 4.1 Investigate recursively defined sequences, including arithmetic and geometric sequences and the Fibonacci sequences.
- 4.9 Construct and use tables to calculate repayments, balance owing and number of repayments on loans and investments. This includes the recognition of patterns to extend the table.
- 4.10 Construct and use tables to compare simple, compound and reducible interest in borrowing and investing money.

At our college, recursion (point 4.1) is covered early in the year, with financial mathematics items addressed late in the year. We have found that the following approach to annuities and loans provided excellent revision of the calculator skills that the students already had. In general, we expect students to step through some of the work by hand first, in order to get a good feel for what is happening. Having drawn up a table, they then look for patterns in the table, and use their knowledge of recursion to write the patterns in a recursive way.

Once they have a good sense of the relationships, we help them to use their calculators in RECUR mode to ease some of the arithmetic. We prefer RECUR mode to the use of the TVM mode partly to emphasise the mathematical processes involved, but also because not all students have a calculator model with TVM.

The next two pages contains a student handout, *Annuities*, suggesting some situations that can be handled using the calculator, after doing them in the table first. The following page extends this method to include loan repayments, in an examination question format.

For task 1, some screens are shown to remind students how the calculator can be set up to handle this kind of situation. Later tasks require students to set up the calculator by themselves.

Reference
Curriculum Council of Western Australia (2001) *Discrete Mathematics syllabus* [http://www.curriculum.wa.edu.au/pages/subject/subject07.htm]

Worksheet 1: Annuities

Annuities questions should be addressed using tables at first. Once you understand the table structure, you can represent the table in RECUR mode on the calculator.

Example 1. Chris decides to invest $400 per year in an account with an interest rate of 8% per annum. Copy and complete the following table.

Year	Amount at Beginning of Year	Interest	Amount at End of Year
1	$400	$32	$432
2	$832		
3			
4			

To see the structure, rewrite the table as

	A	
Year	Amount after Deposit	Amount + Interest
1	400	400(1.08)
2	400(1.08) + 400 = 832	832(1.08)
3	832(1.08) + 400 = 1298.56	1298.56(1.08)
4		
…		

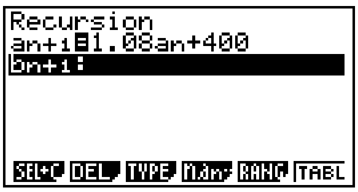
If the balance in year N, column A, is represented as a_n, then the balance in year $N+1$ is a_{n+1}, where $a_{n+1} = Pa_n + Q$

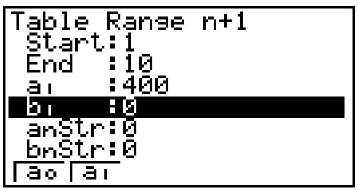
What were the values of P and Q for this example?

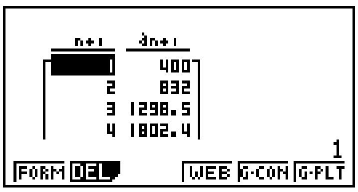
What variables will P and Q always represent?

Use your calculator in RECUR mode, as shown below, to complete the above table, and answer these questions:

(a) check the amount in the account at the beginning of year 5.
(b) check the amount in the account at the end of year 5.
(c) find the amount in the account at the beginning of year 8.
(d) find the amount in the account at the end of year 10.







Example 2. Ben invests $100 every quarter in a bank account earning 12% p.a. compounded quarterly. Copy and complete the following table.

Quarter	Amount at Beginning of Quarter	Interest	Amount at End of Quarter
1	$100	$3	$103
2	$203		
3			
4			

Rewrite the table as for Question 1, and use your calculator to answer the following questions:

(a) check the amount in the account at the beginning of quarter 3.
(b) check the amount in the account at the end of year 1.
(c) find the amount in the account at the beginning of year 3.
(d) find the amount in the account at the end of year 5.

Example 3. On every birthday, Carolyn receives $50 by her husband. On her 35th birthday she decides to invest her money and all future money into an annuity earning 10% p.a. Use your calculator to develop a table as in Questions 1 and 2, and use that table to find:

(a) the amount Carolyn has in the account just before her 40th birthday.
(b) how much she has in the account just after her 50th birthday.

Example 4. Peter needs to save a total of $20 000 for a trip to Europe. He invests in an annuity, which pays 6 % p.a. compounded monthly. He pays $250 every month into the annuity. Fill in the first 4 months of the table showing this.

Month	Balance after Deposit	Balance + Interest
1	250	251.25
2	501.25	
3		
4		

(a) Write the recursive formula for this annuity where a_n is the balance after his n^{th} deposit and a_{n+1} is the next balance.

(b) How long will it take Peter to save the required amount for his trip?

Worksheet 2: Loan repayments

The table below shows Ross's loan account for the purchase of his car.

	A	B	C
Year	Opening Balance	Balance + Interest	Amount after Repayment
1	4000	4000(1.2)	4000(1.2) – 1000
2	3800	3800(1.2)	3800(1.2) – 1000
3			
4			
5			

(a) Enter the values for Year 4.

(b) Calculate the annual interest rate for the loan.

(c) How much is the annual repayment?

If an entry in column A is represented as a_n and the next entry in column A is represented
a_{n+1}, then $a_{n+1} = Qa_n + G$.

(d) Determine from the table the values of Q and G.

You could now produce a relevant table in your calculator.

(e) Enter the values for the last meaningful line of the table. (Year 9.)

(f) Determine the total amount of interest paid on this loan.

Köchel numbers and the age of Mozart

Level
Upper secondary

Mathematical ideas
Linear relationships, residual plot, line of good fit, sampling, predicting

Description and Rationale
Although Wolfgang Amadeus Mozart (27 January, 1756 – 5 December, 1791) is one of the most famous composers of all time, much of his work is undated, and he did not have a publisher. So Dr Köchel, an Austrian botanist and mineralogist, compiled a chronological thematic catalogue of Mozart's works. The catalogue gives each work a number (called a Köchel number), by which the work is now universally identified. Köchel's catalogue was first published in 1862, and has been periodically revised.

Colin Fox is a professional mathematician who also hosts programs on the Australian Broadcasting Commission's FM classical music radio network. He claims that there is a linear relationship between Köchel numbers and Mozart's age when the particular work was written.

This activity allows students to examine this claim informally, and to then use the results of their analysis to predict how old Mozart was when a composition with a particular Köchel number was written.

Taking a sample
Mozart was a prolific composer over his short life, and began composing as a young child prodigy. The Köchel numbers range from 1 to 626, suggesting that 626 works have been attributed to him. Rather than try to deal with the entire collection of Mozart's works, the claims for a linear relationship can be informally scrutinised with a sample of data. To do this with the calculator, select at random a set of numbers between 1 and 626 and use the corresponding pieces of music for an analysis.

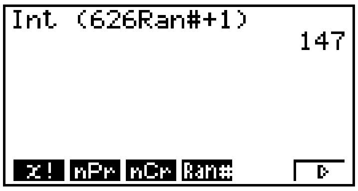
Köchel's numbers have been revised from time to time, as musicologists reconsider the status and composition date of particular pieces of music and even find new pieces of music now attributed to Mozart, but not originally recognised by Köchel. Köchel numbers are now in their sixth edition, and later editions have not always used integers, so that taking samples using integers is a little problematic. However, for the purposes of informally examining the relationship between catalogue number and age of the composer, a sample from the original catalogue is adequate.

There are a number of websites containing the complete catalogue. One useful one that was used to illustrate this activity is at the URL:

http://www.classical.net/music/composer/works/Mozart

This website, like many others, is protected by copyright, so the contents have not been reproduced here.

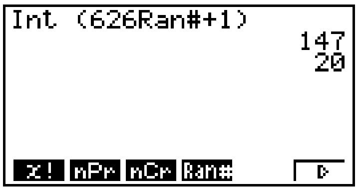
To choose at random an integer between 1 and 626 inclusive, a suitable calculator command is shown in the screen below. The Int command is available in the NUM menu and the Ran# command is available in the PROB menu. These two menus are both accessible via the OPTN key.



In this case, the random integer generated was 147.

The composition with Köchel number 147 can be determined from the website as the song, *Wie unglucklich bin ich nit*, thought to have been composed by W. A. Mozart in the period 1775-1776.

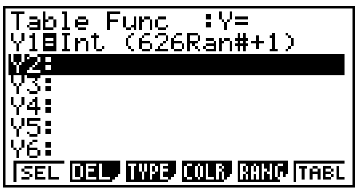
Pressing the EXE key a second time repeats the same command, thus selecting at random a second composition. In this case, it is Mozart's Motet in G Minor (*God is our refuge*), K.20, composed in July 1765.

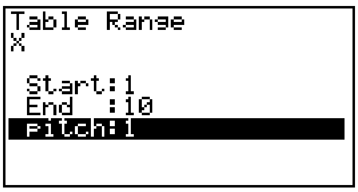


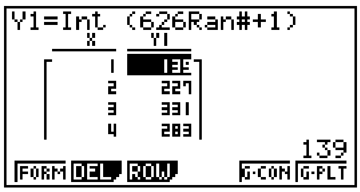
Students could continue in this way, gathering a sample of data to use to consider Fox's claims. They will notice, as with these two examples, that there is some variation of precision of dates of composition. Some are given with a particular date, while others are more speculative, and much less precise, such as the first one chosen above. Sampling real data in this way will help students realise the need for caution when conducting practical work of this kind.

A more efficient mechanism

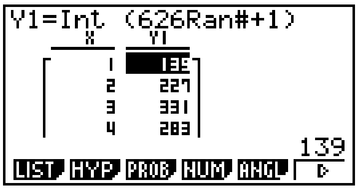
A more efficient mechanism for taking a random sample of Köchel numbers is to use the TABLE mode. Before starting, make sure that the STAT lists have been cleared, so that data can be transferred into them. The screens below show how a sample of ten numbers was chosen.

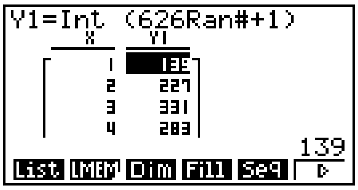


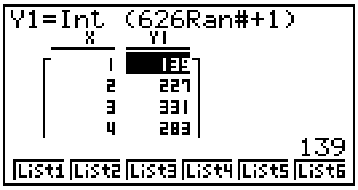




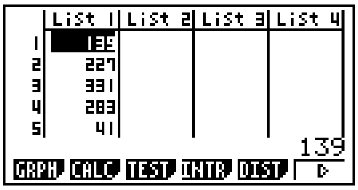
The resulting sample can be transferred across to the STAT mode of the calculator for analysis by positioning the cursor in the Y1 column of the table and using the LMEM (List memory) command available by first choosing OPTN and LIST. The next screens illustrate the steps in this process.







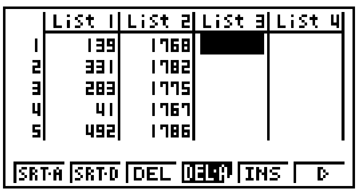
After List 1 was selected (with F1), you can check the transfer process by moving to STAT mode. The screen below shows the first few elements of the transferred data. Notice that the randomly chosen Köchel numbers match those generated in the table.



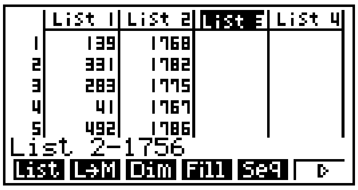
Once these are chosen, the website can be consulted to find the dates of composition. For the purposes of this illustration, the dates have been represented as a single year. (in some cases, this involves collapsing a range of years to a single year, thus making a further approximation.) Here is the complete sample we obtained on this occasion, using the original Köchel numbers as the source of data:

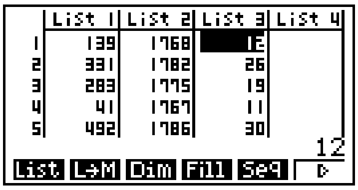
K	**Composition**	**Year**
139	Missa Solemnis in C Minor, *Waisenhaus*	1768
227		
331	Sonata in A for Keyboard	1782
283	Sonata in G for Keyboard	1775
41	Concerto in G for Piano, Number 4	1767
492	*le nozze di Figaro*	1786
43	Symphony in F, Number 6	1767
556	Canon in G for 4 voices in 1, *Grechtelt's enk*	1788
465	Quartet in C for strings, *Dissonant*	1785
210	Aria for Tenor, *Con essequio, con rispetto*	1775

In this case, one of the works chosen (K227) appears to have no corresponding composition in the list given on the website, partly as a consequence of using the original Köchel numbers, or possibly as an error in compiling the website. So in this case, the sample comprises only nine elements rather than the intended ten, and the number is removed from the data set. The next screen shows the dates added (individually) as List 2 to the data set.



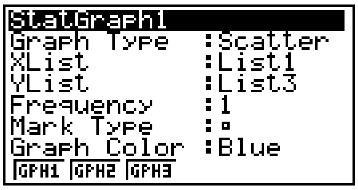
To determine the age of the composer for each of these works, you can subtract 1756 (his birth year) from each and record the result in List 3. A powerful alternative is to use a List transformation as shown below. After placing the cursor on the list name the OPTN button and LIST(F1) menu show the commands available for list operations.

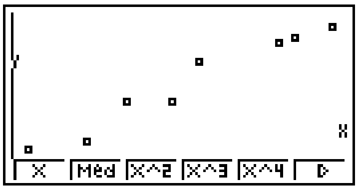




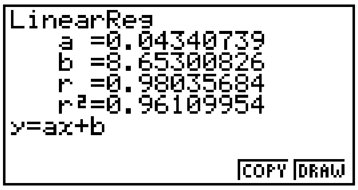
Analysing the data

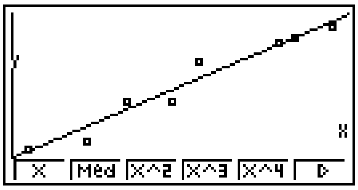
The easiest way to examine the claimed relationship for this particular sample of data is to draw a scatter plot, with Köchel numbers (List 1) on the horizontal axis and Mozart's age (List 3) on the vertical axis. The screen below shows the relevant GRAPH settings in the STAT menu in this case, together with the resulting scatter plot (using the AUTO setting for the Statistics window).





The scatter plot has a strong suggestion of linearity, and a suitable line of good fit can be estimated using the X command (F1) as shown below.





For this (small) sample, the line of good fit provided by the calculator for predicting Mozart's age from the Köchel number is approximately

$$age \approx 0.04 \times K + 8.7$$

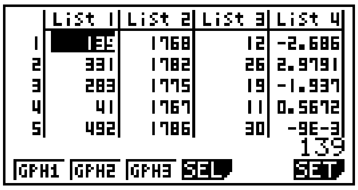
This rule of thumb can be tested on a second random sample of compositions.

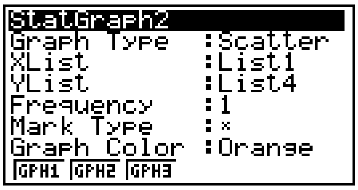
Students can be asked to devise a rule of thumb from their data and use it to validate their model. Rules of thumb should be fairly easy to use and hopefully reasonably accurate, although necessarily approximate. In this case, a possible rule of thumb might be

Divide the Köchel number by 25, then add 9.

As well as using inspection of the closeness of the line to the data, students can examine the fit of the linear model to the data using a residual plot. To do this, they will first need to instruct the calculator to automatically compute residual and to store them in an empty list (a good choice in this case is List 4). The corresponding SET UP commands are shown in the next screens.

After this setting has been made, residuals are automatically computed when a model is chosen, and then are stored into List 4. In this case, a plot of the residuals suggests that they are random in character, giving some support to the validity of the linear model:





Comparisons and use of rules

It is a useful classroom activity to compare the rules of thumb obtained by different students and by the same students on different occasions. You might also choose to reveal to students at a late stage that Colin Fox's rule of thumb is:

Divide the Köchel number by 25, then add 10

In this case, the rule of thumb devised with a small sample of data is quite similar to Fox's rule.

Students might also be encouraged to speculate on how many compositions Mozart might have produced if he had not died at such a young age.

This would of course require them to think about the assumptions involved, especially the assumption that he would continue to generate compositions at the same rate throughout his life.

If Mozart had lived to 70, what might have been the Köchel number of his final composition?

Reference

Classical net (1995) *Köchel's catalog of Mozart's works*. [http://www.classical.net/music/composer/works/Mozart]

Parabolic motion reborn

Level
Upper secondary

Mathematical Ideas
Parametric equations and the application of mathematics

Description and Rationale
This lesson leads students to compare the properties of parabolic motion when the initial conditions of flight are varied. The graphics calculator provides an efficient medium to simulate projectile motion once the students understand the basic concepts of projectile motion. This is a very powerful feature and allows students to see what is happening and get a genuine feel of the concept of time of flight. It also offers tools to help solve problems that have traditionally solved by mind, paper and pen. It is left to the reader to consider the value of this approach – mind and calculator.

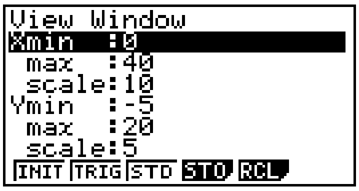
Students can investigate how the height above ground at a given time, maximum height reached and range of the object depend on the angle of projection.
Consider a projectile that is launched at 20 metres per second at 30, 45 and 60 degrees above the horizontal.

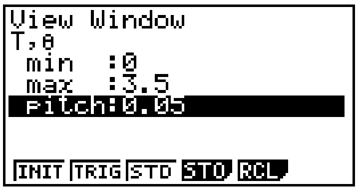
In the first of these cases the students should find that the components of the motion can be described by the pair of parametric equations, $v_h = 10t\sqrt{3}$ and $v_v = 10t - 5t^2$

Now, in graph mode the students can simulate the motion and leave a trail for analysis.

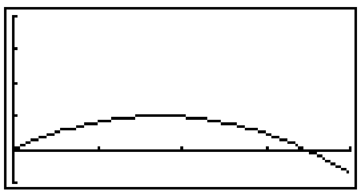
Once the graph type is changed to parametric (using TYPE (F3)) the equations can be entered.

Appropriate settings for the view window are seen below.

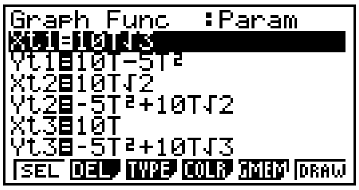


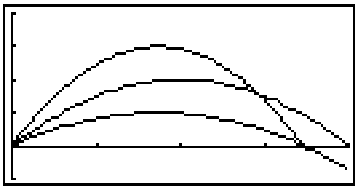


Drawing the graph (using DRAW (F6)) reveals the path seen opposite.

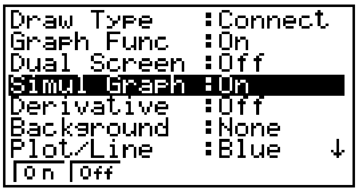


Entering the other two sets of parametric equations that describe the paths being investigated will result in the graph seen below.

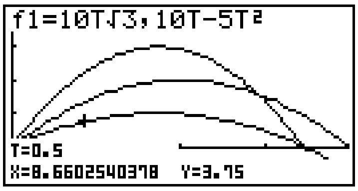


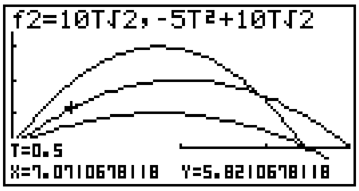


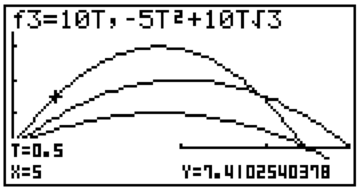
If you enter the SET UP of the graph mode (SHIFT then MENU) and turn the Simul Graph option on, the projectiles will be launched in unison. This provides the viewer with an appreciation of the time of flight of the projectiles. This is a great advantage to students in their quest to understand projectile motion.



While the graph is on screen, students can use the TRACE (F1) facility to explore the path and compare the features of each path. Using the up and down arrow will allow you to swap between paths.



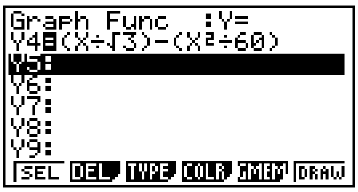


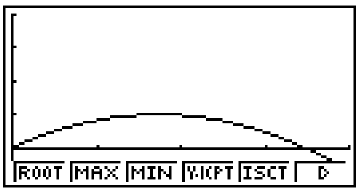


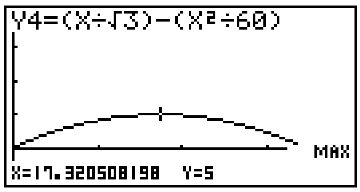
Note how the cursor simulates the projectile and you can see where each object was a 0.5 of a second after launch. Also given are the horizontal and vertical positions at this instance.

Students can tabulate values of interest for comparison purposes. It may or may not be possible to trace to the peak of flight and point of impact with the present settings for time (T). The challenge still exists for the students to find these points. However, one may ask whether the level of accuracy we once required is important. It is left to the reader to consider this. But, please read on.

When drawing the graph of a set of parametric equations the treasure trove of tools in the G-Solv (F5) menu does not work. Students could, however, substitute to get the vertical height in terms of the horizontal distance travelled, change the TYPE of graph to a Y=, and then use the MAX and ROOT within G-Solv tools as seen in the screens below.







Many possible investigations can now take place. Students could investigate the effects of changing the launch height and launch speed.

In addition to this students may use the calculator as a tool to aid in the solution to problems like:

> *A plane flying at 300m above the ground at 150 metres per second releases a food parcel at a point that is 800m (horizontally) from the target. Did the pilot calculate well? If not determine the point at which the parcel should have been released.*

It is left to the reader to consider whether or not the use of the calculator to aid in the solution of such problems is desirable.

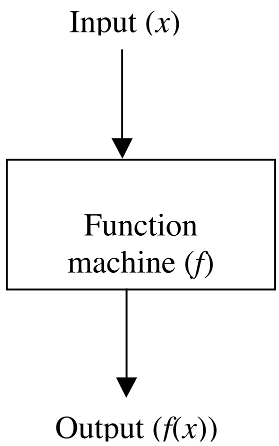
Composite functions

Level
Upper secondary

Mathematical Ideas
Function, composition of functions

Description and Rationale
The composition of functions can be a difficult concept for students to understand. Student understanding may be deepened and consolidated by them experiencing a numeric, graphical and algebraic representation of the concept. The graphics calculator is of great aid in this pursuit. Traditionally only the algebraic representation is presented to students.

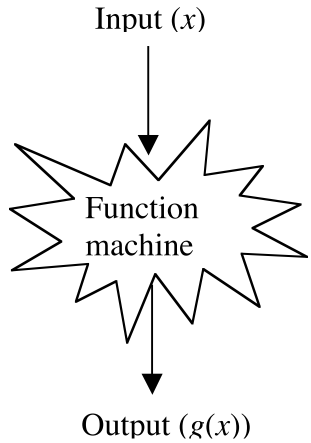
It is suggested that students begin their learning of this concept without using a graphics calculator. It is important that students understand the simply idea at the heart of this concept. The idea of a *function machine* is a nice way to start. This is something that can be developed in the earlier years of school. A function machine works as follows: a number is placed into the machine – the *input*, operated on in some manner and then the new number exits the machine – the *output*. If we let the input be x and the function machine *subtract five* to the input then the output will be x - 5. The output is a function of x and hence are denoted as $f(x)$. This can be displayed pictorially as shown opposite.



To develop a numerical understanding students should complete a table of values as follows:

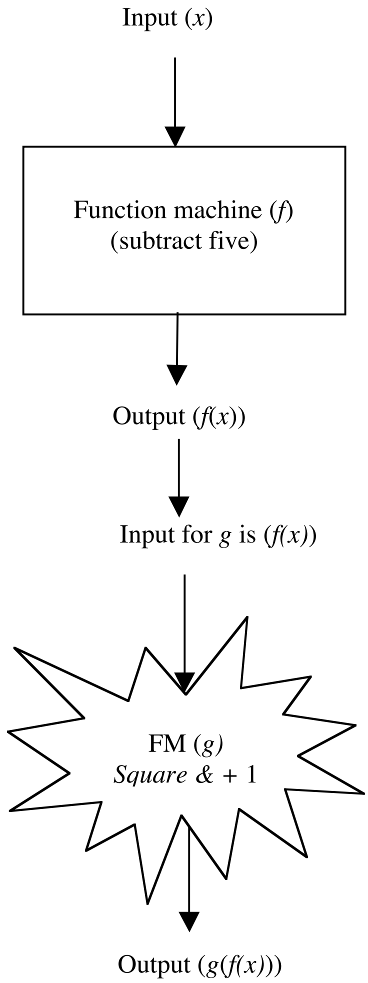
x	***f(x)***
4	-1
5	0
6	1
7	2

Let's now consider a second but different function machine g that *squares the input and adds one*. If the input is again x then the output will be a function of x denoted $g(x)$.
A table could then be formed.



x	$g(x)$
4	17
5	26
6	37
7	50

The idea of the composition of two functions can now be explained as the joining together of these two machines, end on end so that the output of one becomes the input of the other. If we let the output of f, $f(x)$ become the input of g, then the output of g, can be denoted $g(f(x))$. We can develop the following picture and table.



x	$f(x)$	$g(f(x))$
4	-1	2
5	0	1
6	1	2
7	2	5

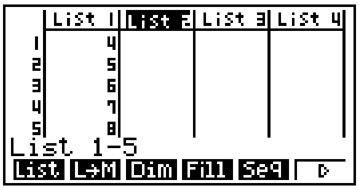
It is likely that the students will have developed the third column of this table from the second column, following the idea of the output of f being the input of g.

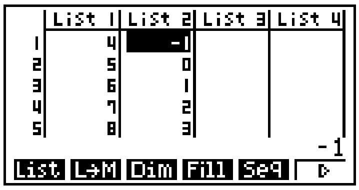
It is now that the students should be challenged to develop the third column directly using some set of operations on the x values.

After having done a few examples by hand the graphics calculator comes in very handy here. The students can use the lists in STAT mode to achieve this end.

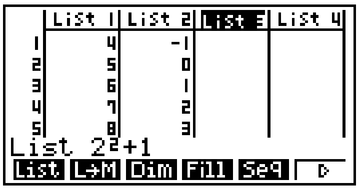
You will see that this gives a lovely conduit between the numeric representation and the algebraic representation.

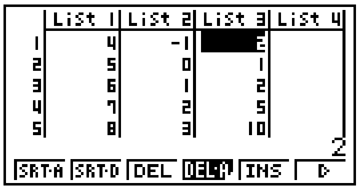
With the cursor in the list name values of *f* can be generated as shown below, using commands found via the OPTN button and LIST(F1) menu.



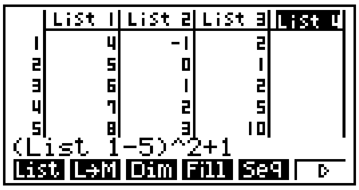


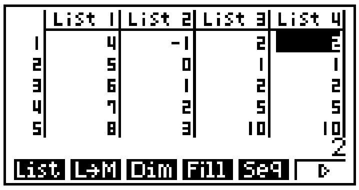
The values for *g* could then be produced as follows:





or, in List 4 as follows, the lists 3 and 4 can then be compared:





Note that these tasks are leading the student very nicely toward the algebraic representation for $g(f(x)$. Using the List gets us one step closer to the generalisation than does the work done on paper as the student works on one cell of the table at a time.

Students could then be asked to work with different pairs of functions in a similar way to consolidate their understanding of this process. A first example could be to reverse the order of the function machines we have been using to see if $g(f(x) = f(g(x)$.

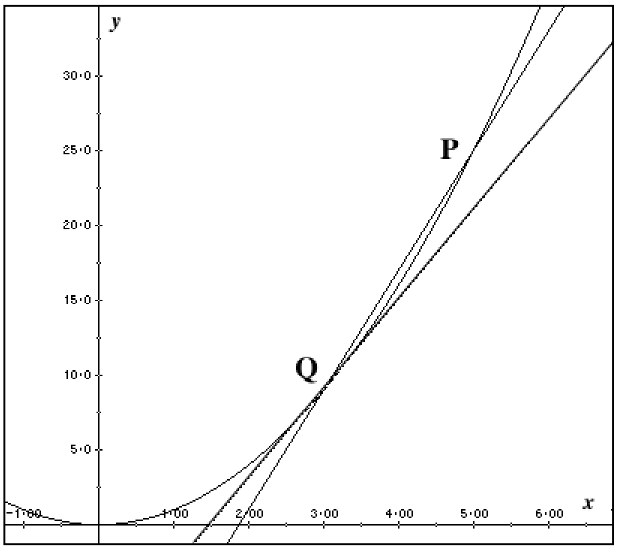
At this point students should move to the algebraic representation of this concept for the cases studied. Once comfortable they could move into the more traditional type questions that involve proving certain results.

Slope of a tangent

Level
Upper secondary

Mathematical Ideas
Slope of a tangent, limits

Description and Rationale
This lesson acts as an introduction to the concept of a limit and differential calculus. The graphics calculator allows students to perform and *display* repeated calculations with ease, which is of great advantage in this task.



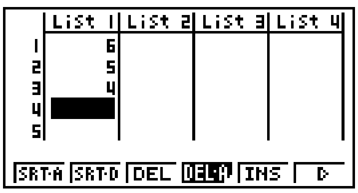
Students are asked to consider the problem of determining the slope of the tangent to the curve $y = x^2$ at $x = 3$. They would be aware that they can simply determine the slope of any chord (like the slope of PQ) on this curve, but the tangent is another matter.

Once the students appreciate that we could consider P as a moving point and move it toward the fixed point Q they will see that we may have a way to approximate the slope of the tangent. That is, make the chord become as close to a tangent as possible. Dynamic geometry packages offer the lovely option of being able to 'drag' the chord toward the point of tangency.

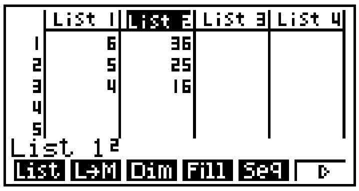
Students should now produce a table, using mind, paper and pen, illustrating the slopes of the chord at 3 different points along the curve as P approaches Q. For example when $x = 6.5$ and 4. They should be encouraged to consider the situation when P maps to Q.

We can use the graphics calculator to efficiently investigate how the slope of chord PQ varies as P moves closer to Q at many more points than four and at much smaller increments than would otherwise be engaging and hence, see whether or not a reasonable approximation of the slope of the tangent at Q can be found. The lists in the calculator can be used to do this.

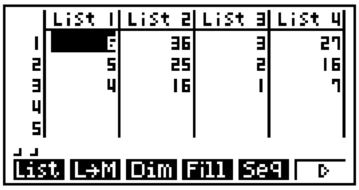
In LIST or STAT mode we can type in the required values.

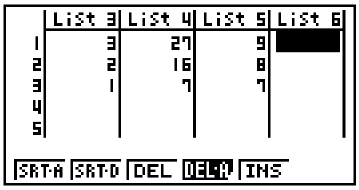


In List 2 we can calculate the value x^2 as shown opposite. This is completed by highlighting the list name, selecting OPTN and LIST (F1) and then squaring the values in list 1 as shown.



In List 3 we can calculate the *x* step (or run), in List 4 the *y* step (or rise) and then in List 5 the slope of the respective chords as seen below.

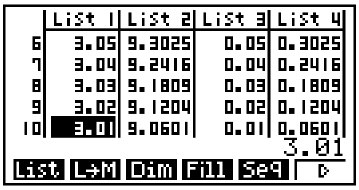


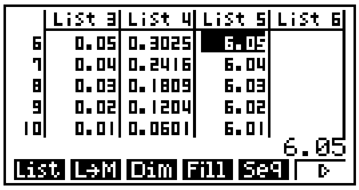


This should replicate what the students have done with mind, paper and pen. It acts as a nice check and simple way to get started.

Now students can begin to experiment with smaller increments between chords, and chords that are closer to being tangential.

The screens below show one possible output.





Care will be required when determining how many elements are generated in List 1. It has a limit of 255 elements and the calculator will return a memory error if you try to exceed this.

Discussion can then follow this activity about the value of the slope of the tangent that seems sensible and the concept of a limiting value.

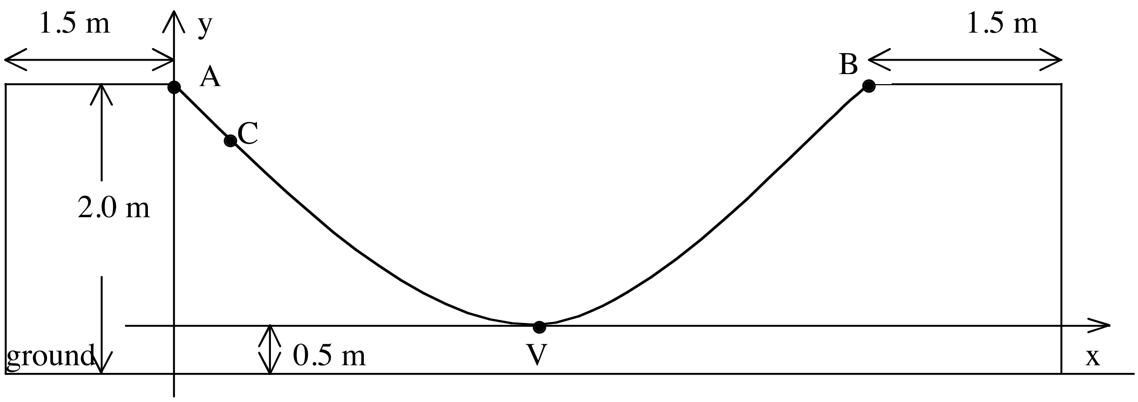
Skateboard ramp

Level
Upper secondary

Mathematical Ideas
Quadratic functions

Description and Rationale
This activity has students experimenting with different quadratic functions to attempt to form a suitable shape for a skateboard ramp. The graphics calculator offers the students an efficient way of displaying a number of quadratic functions until they are satisfied while reinforcing their knowledge of the effects of changing the coefficients of quadratic functions. The task, as written for students is offered below.

The diagram below is a cross-section (side view) of a skateboard ramp.



The aim of this investigation is to produce a quadratic function that models the shape of the ramp from point A through V and up to point B. The x and y axes have been included in the diagram to assist your model development. The top of the structure is 2 metres above ground level. The vertex, V, of the parabola is 0.5 m above ground level. The width of the ramp is dependent on your choice of model and will become known through your working.

There are three parts to this investigation.
Initially, you will set up a general equation for the parabola, then modification of this model takes place by considering what shape and width you desire for your ramp. Finally, you refine your model by considering how steep you want the drop to be from point A.

Part One

The curved part of the cross-section shown above is the shape of a parabola, and can be modelled by the quadratic function:

$$y = ax^2 + bx + c.$$

The vertex is labelled V on the diagram.

1. Using your knowledge of quadratic functions, what are the coordinates of V?
2. Hence, show that $c = \frac{b^2}{4a}$.
3. What are the coordinates of A?
4. Hence show that $c = 1.5$, and that the quadratic becomes $y = \left(\frac{b^2}{6}\right)x^2 + bx + 1.5$.
5. Use the symmetry properties of quadratics to find an expression for the coordinates of B in terms of b.
6. Revise your coordinates for V from question 1 so that any expressions are in terms of b.
7. Write down your equation for y from question 4, and the points A, B and V in terms of b.

Part Two

Your aim is to model the skateboard ramp surface by altering the function y obtained in part one. You should check the suitability of the equation by altering the value of b to produce suitable positions for B and V. For instance, how far apart do you think A and B should be? Continue through part two and you will answer these questions.

1. Given the information you have produced in question 7 above, and looking at the diagram, what is the sign of b? State is your reasoning for this decision?
2. Choose three values for b which you think may be suitable for this model. [Hint: consider the choice of b and the coordinates of V and B]
3. For each value of b, define the function y in GRAPH mode of your graphics calculator.
4. You may prefer to plot all on the same set of axes, so use the colour choices to help distinguish between each curve.

Before drawing the curves, ensure that your view window has sensible settings for your situation. [Look at the diagram on page 1 to help you.]
To give you a more realistic idea of the shape of your curves select SHIFT F2 (ZOOM), then F6 to see further options, then F2 (SQR). This will give you a view of the axes so that 1 unit on the x axis is equivalent to 1 unit on the y axis.
Use the ZOOM to organize the axes so that you can see which curves model the ramp to your liking.

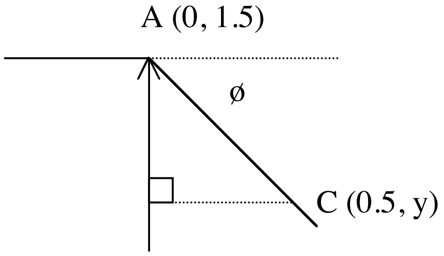
If you need to investigate further values of b do so.
Remember to check the distance between A and B. Is the distance reasonable? If not, choose a different b value until you are satisfied.

Choose three of your “best” b values and discuss and sketch the shape of your skateboard ramp model.
Now choose one of these values of b that models the skateboard ramp to your liking and record it. Write the equation for y, and the points for A,V and B.

Part Three
Further refinement of the model:

To refine your model you will investigate how steep you want the surface to be from A.

Look at the diagram on page 1. The point C is a small distance away from A.
Let us say the x coordinate of C is 0.5.
Find the y coordinate of C using your equation for y chosen in question 5 (part two).
A close up of the region near A and C looks like this:



Assume that AC is close to being a straight line. Then angle ø is the angle of descent.
Use right-angled trigonometry to calculate ø to the nearest degree for your model chosen in part two question 5.
If this is not steep enough (the angle of descent should be larger than 50˚) try another b value from Part two question 4 and record your calculations for ø. Keep doing this until you are satisfied with your model.

The final model
Write down your model for the cross-section of the skateboard ramp, including the coordinates of A, B and V, and the angle of descent from point A.

Draw a scale diagram of your model.

Comment on any parts of the ramp that may still be inappropriate. Suggest another type of function that may be useful to model a skateboard ramp.

Integration – using the ‘area so far’ function

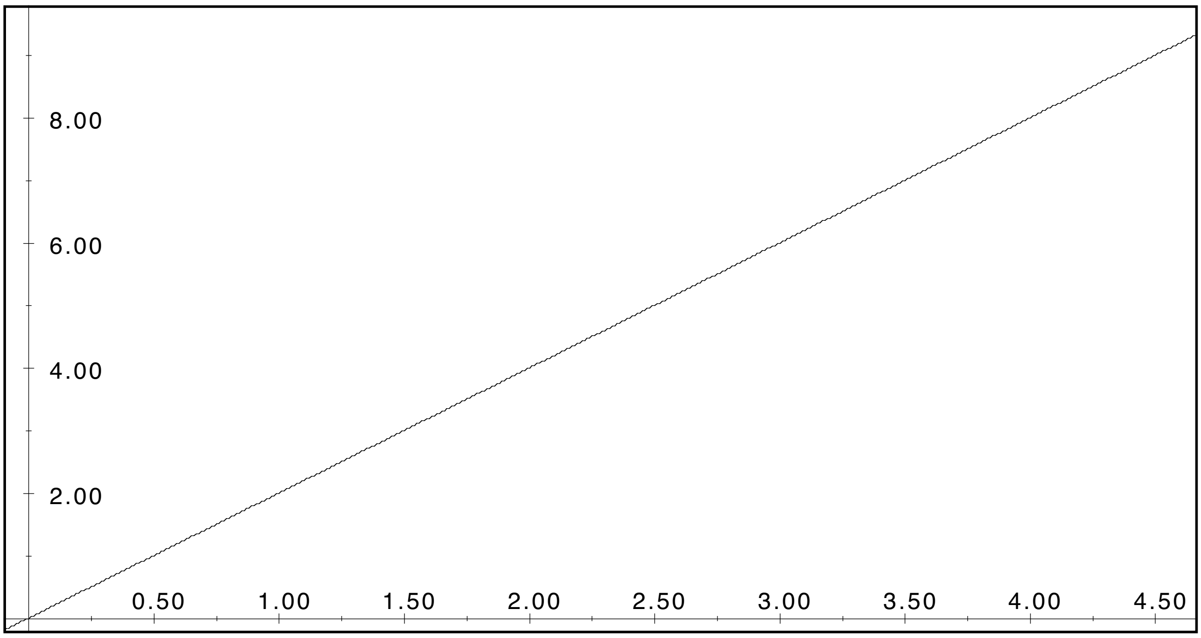
Level
Upper secondary

Mathematical Ideas
Area under a curve, integration, collecting and modelling data

Description and Rationale
The area under a curve can be a very useful mathematical tool. When students begin studying integral calculus methods such as the trapezoidal, the Monte Carlo and upper and lower rectangle methods are used to determine the area under a curve. These are estimates that can be very accurate under the right conditions (large sample, small increments etc..) but they are still estimates and thus do not provide exact answers. Further they are often cumbersome and time consuming in their calculation and do not provide general relationships. This activity attempts to explore the possibility of finding a mathematical relationship between a function and the area under the curve defined by the function.

One way of investigating the relationship between two quantities is to collect some information, make a conjecture and then check its validity. To do this it is best to start with a very simple example. A linear function is such an example.

Consider the function y = 2x, the graph of which appears below.



Students would be asked to investigate the area under this curve upward from x = 0.

Students determine the area under the curve between x = 0 and a range of x–values greater than zero, this information is then recorded and modelled.

Example

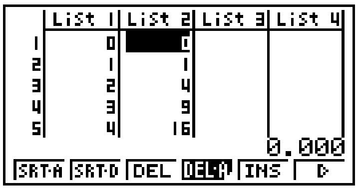
Between $x = 0$ and $x = 2$ $\qquad A = \frac{1}{2} \cdot 2 \cdot 4 = 4$ square units.

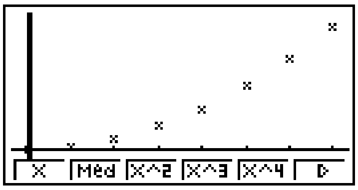
Between $x = 0$ and $x = 4$ $\qquad A = \frac{1}{2} \times 4 \times 8 = 16$ square units.

A sample of the possible data that may be generated is recorded in the table below.

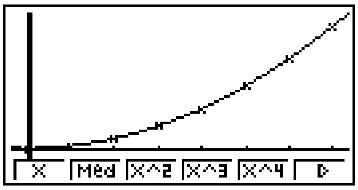
X	0	1	2	3	4	5	6	7
AREA	0	1	4	9	16	25	36	49

This data can then be placed into lists and a scatter plot produced.





Students would then be expected to model this data in an appropriate way. Students may observe the data collected and see the model would be "area so far" = x^2. Others may note the shape and predict a quadratic model. A third method involves using the built in quadratic regression modelling tool.

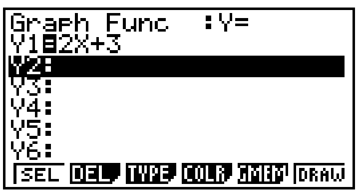


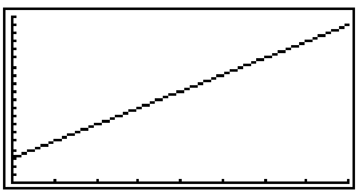
Based on the information generated above students would be expected to produce a conjecture along the lines of:

"the area under the curve from x = 0" function for the function $y = 2x$ is $A = x^2$.

The area under a curve can easily be determined using the calculator in either the GRAPH mode (using the commands in G-solv) or the RUN mode (using commands in OPTN and VARS).

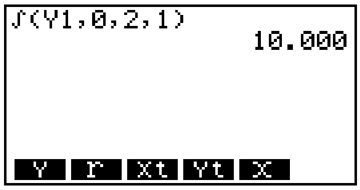
Examples of each, for the function $y = 2x + 3$, are shown below.



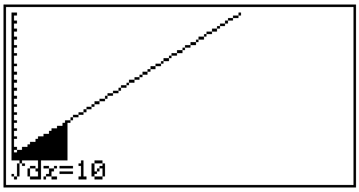


The examples below find the area between $x = 0$ and $x = 2$ for $y = 2x + 3$

In the RUN mode the integral sign can be found under OPTN and CALC (F4). The function Y can be found using VARS and GRPH (F4). The lower and upper terminals are defined along with a number which relates to the tolerance method employed by the calculator, 1 is the most accurate.



In the GRAPH mode the integral sign is found in the G-Solv menu. Sideways arrows and EXE are used to define the lower and upper limits.

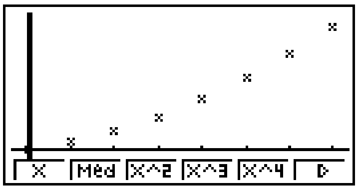


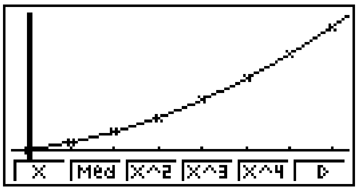
Students may then be asked to use one of the methods above to collect data and model the rule for "the area under the curve so far up from $x = 0$" for the function $y = 2x+3$.

A possible data set that may be generated by the students is shown below.

X	0	1	2	3	4	5	6	7
AREA	0	4	10	18	28	40	54	70

The resulting scatter plot and model could be determined using one of the methods described above. The model is $y = x^2 + 3x$.





So the "area under the curve so far from x=0" for the function $y = 2x+3$ is $A = x^2+3x$.

Students would then be asked to make a general conjecture about the relationship between these two functions. It is hoped that they would begin to see the idea of anti-differentiation of polynomials as they would previously have studied the rules for differentiation.
To verify the conjecture members of the class could each be given a function to investigate, with the results summarised in a table such as shown below. This may generate much class discussion and collaboration between students as they develop and discover the relationship between the functions. The examples chosen, at this stage, must have areas that are restricted to being above the x- axis.

Group results

Original function	"Area so far" function
$2x + 1$	$x^2 + x$
$6x + 4$	$3x^2 + 4x$
$-3x + 30$	$-1.5x^2 + 30x$

Beyond Linear functions

As we know mathematical models are not limited to just linear functions.
The motion of a body thrown upwards under gravity produces a parabolic position-time graph. The mathematical representation of this is:

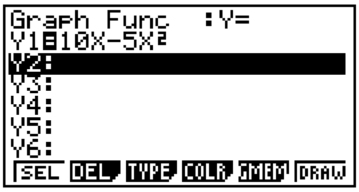
$$s = ut - \frac{1}{2}gt^2$$

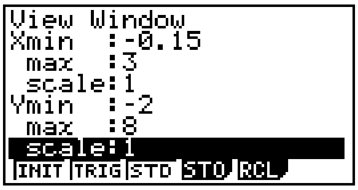
where s is the position (m), u is the initial velocity (m/s), t is the time (s) and g is the acceleration due to gravity (m/s^2). We need to check whether our conjecture from our work with linear functions still holds with a more complex function.

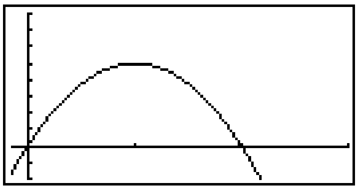
Consider a body thrown upwards at 10m/s (u), assuming that gravity (g) is 10m/s^2.

The position is given by $s = 10t - 5t^2$

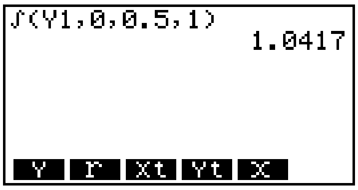
Students could be asked to produce a plot, collect data relating to the "area so far" for a range of x-values and then complete an analysis as before. Students should be encouraged to take into account the conclusions drawn from the linear case.
The view window is chosen to allow x values for the integral to easily be selected. The window uses the fact that the screen is 126 pixels wide.

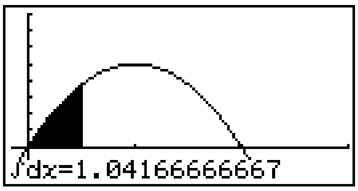






The examples below involve choosing a lower bound of 0 and an upper bound of 0.5.

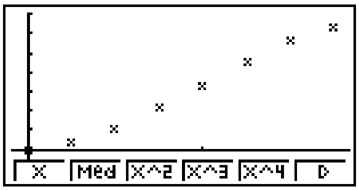


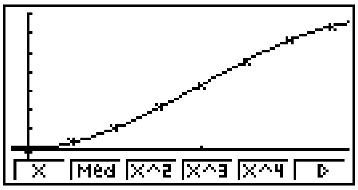


A set of data, such as that shown below, should be generated.

X	0	0.25	0.50	0.75	1.00	1.25	1.50	1.75
AREA	0	0.287	1.042	2.109	3.333	4.557	5.625	6.380

The scatter plot suggests a cubic function, using the ideas explored previously a model must be fitted.





Students would be expected to explain clearly how the information gained from the linear cases relates to that gained from the more complex function. A good student should be able to clearly articulate the development of their understanding of the "area so far from x = 0" function.

Students understanding could be quickly checked by a series of examples set out in a table such as that below.

Use what you have learned to write down the "area so far from x = 0" functions for each of the following:

Original function	"Area so far from x = 0" function
$y = 3x - 2$	
$y = -3x^2 + 2x$	
$y = 4x - x^2$	
$y = 2x^2 – 3x+ 4$	
$y = x^2 + 2$	

Limitations

You have found a rule for the "area so far" for most polynomials. This process is known as integration and the "area under the curve" function is known as the integral. You may also have noticed that this is the same as deriving backwards. For this reason this process is often called anti-differentiation and the resulting function called the anti-derivative.

This rule has proved fairly effective for a range of functions but we need to be aware of its limitations.

Students may have noticed that we have only investigated the "area so far' functions that start at $x = 0$. What alterations to this method do we need to make to find the area under a curve between two non-zero values, eg the area under the curve $s = 10t - 5t^2$ between $t = 2$ and $t = 4$?

Students could be asked to investigate this problem and present their findings to the rest of the class. They would be expected to write down what they have found and provide evidence that their solution is correct.

A further extension would be to investigate situations where the graph crosses the axis within the domain explored. The ideas of positive and negative areas and the separation of a function into regions for determining the area would then be explored.

Data logging and cooling curves

Level
Upper secondary

Mathematical Ideas
Data logging, exponential functions, transformation of functions, mathematical modelling

Description and Rationale
Along with the advent of the graphics calculator and data collecting devices has come the ability to quickly, easily and cheaply design experiments that provide real data on which students can complete real life mathematical modelling tasks.

Much of the mathematics presently done in schools is removed from the real world and as such seen as not very relevant, with the data logger the student moves closer to the real world applications of mathematics. Students are motivated by the experience of collecting the information, analysing the data and reporting their findings. The opportunities for collaborative learning to occur are immense, with benefits for both the students and the teacher.

Hot water, placed in a cup, cools over time to room temperature. The rate of cooling depends on a number of factors, including the material the cup is made from. The cooling process itself is an interesting one to try and describe mathematically as a number of different types of functions can be developed that model cooling behaviour. This activity will explore the cooling behaviour of a probe in air alone so that the rate of cooling can be observed with a minimum of other factors that may effect the outcome. The cooling behaviour of other materials, under different conditions, may be investigated in follow up activities.

There are many methods and programs for collecting the data in an experiment such as this. GETDATA2 is a program written to control probes on the data logger from the calculator, it is available from the Casio website and is commonly used to run data loggers.

Equipment
You will need the following for this experiment:

coffee cup and hot water
graphics calculator
data logger and temperature probe
thermometer

The Task
Each group is asked to perform the experiment set out below to collect the required data. Each group is responsible for collecting their own data. Room temperature is recorded and then the variation of temperature with time is recorded after the probe is removed from the hot water.

Students may be asked to determine a model for the cooling, interpret the constants of the model and present their findings to the rest of the class

The Experiment

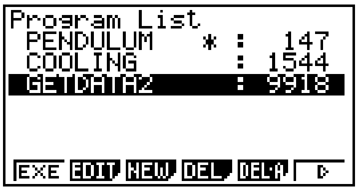
1. Connect your calculator to the data logger via the communication port and the temperature probe to Channel 1 on the data logger.

2. Record the room temperature

3. Fill your cup with boiling water

4. Place the Temperature probe in the water for approximately 2 minutes.

5. Go to the Program menu and choose GETDATA2.



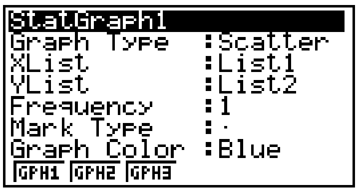
6. We need to set the probe type and the channel it is connected to (SetP) along with the number of samples and the time interval between samples (SetS). These need to be set according to the task being completed. Here we used a temperature probe (Temp) in CH1 and take 90 samples at 1 second intervals.

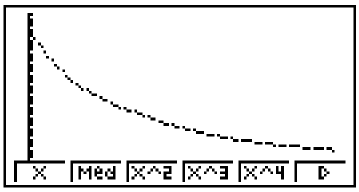
7. Remove the probe and press GO (F1)

8. When the data is collected transfer the data to the calculator, following the instructions on the calculator, observe the scatter plot of the data and attempt to model the data.

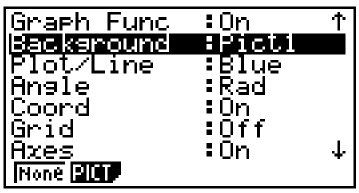
Finding the model

Produce a scatter plot of the data.





So that the data may be modelled a picture is placed in the background. This is done by choosing OPTN and then following the prompts to store a picture. The picture is then placed in the background via the SET-UP (SHIFT then MENU).



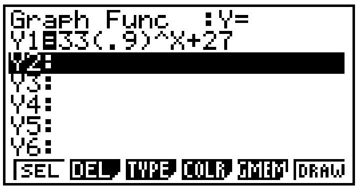
Students then use their knowledge of exponential functions and the transformation of functions to develop and refine a model that they believe fits the data to a satisfactory level. The room temperature was found to be approximately 27°for this data set.

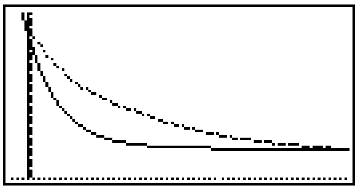
The basic form of one function that may fit the data is $T = a(b)^t + c$ where:

> *a* is the distance between the highest value and the lowest value (room temperature).
> *b* is the decay rate (a value between 0 and 1).
> *c* is the room temperature (the vertical transformation).

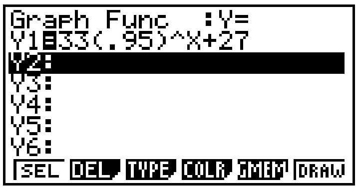
Based on the data collected a value for *a* in the initial model would be (60 – 27) = 33.
A value for *c* would be 27
b could be determined using the common ratio between successive terms or a value could be chosen and then adjusted as necessary, here we will try 0.90.

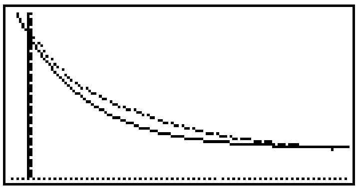
The function can be placed in the calculator and graphed.



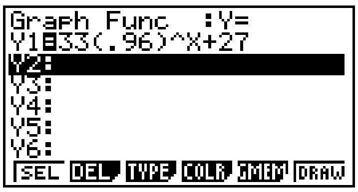


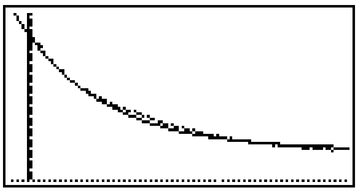
The function is appropriate at the beginning and end of the data but has dropped off too quickly in between. Students would be expected to realise that the decay rate is causing this difficulty and so a value closer to 1 needs to be used. Students explaining these ideas should be an integral component of the report to the class.





With a slight alteration in the decay rate a function that satisfactorily fits the data set can be achieved.





The more conventional expression of the exponential model is

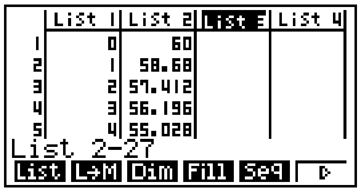
$$y = Ae^{kx} + c$$

$$\text{where } b = e^k, \qquad (k = \ln b)$$

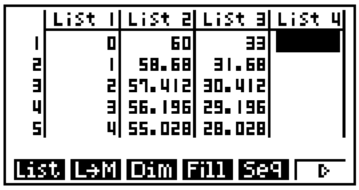
Using this model a result similar to that above could be generated.

Students may have also considered transforming the original data so that a simpler model may be fitted. With the data in the lists a new list, with the temperature lowered by the room temperature can be created.

To complete an operation on a list of numbers the list name must be highlighted. The available commands can be found via the OPTN button, then LIST menu. Here the room temperature is being subtracted from each recorded temperature.



The data in List 3 could now be modelled against List 1 to produce a model without the vertical transformation.



An activity such as this is an ideal opportunity for students to work collaboratively and to communicate how they developed their model, including the reasoning for the adjustments made along the way. Students should be encouraged to communicate, both orally and in written form, the mathematics involved in completing this activity. Some guidelines as to what is expected or required in the students' presentation is desirable for all parties concerned.

While only an exponential model has been investigated here the consideration of other types of models and the reasons for accepting or rejecting them could be an integral component of this activity. The exploration of the limitations of any models investigated is a valid component of any mathematical modelling activity.

Exploring derivatives of trigonometric functions

Level
Upper secondary

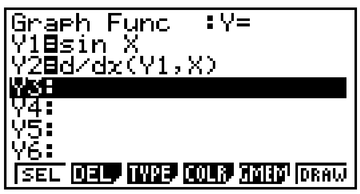
Mathematical Ideas
Trigonometric functions, graphical representations, derivatives

Description and Rationale
The graphics calculator is an ideal device for the structured investigation of patterns in mathematics. One of the more difficult concepts in many mathematics programs involve the relationships between functions and their derivatives. Students often gain a better understanding of these relationships through pictorial representations. The example shown below uses a “black box” approach to the determination of the derivative. This is a valid approach only once students are familiar with the basic definitions and interpretations involved in determining the derivative of a function from first principles. While this example involves trigonometric functions the general approach could be used with a variety of functions.

This structured investigation is designed for students who are not aware of the rules for the derivatives of y = Asin kx and y = Acos kx or who may need to revisit these concepts.

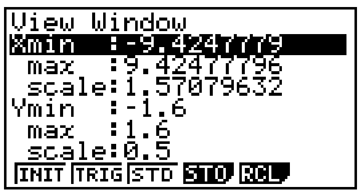
The calculator is set so that a function and its derivative are plotted simultaneously. Color can be used quite powerfully to distinguish between the function and its derivative. In the example below sin$x$ has placed in Y1, while the commands stated below define Y2 to always be the derivative of Y1.

To define the derivative the OPTN button then the CALC (F2) menu is used. The defining of the function Y1 is via the VARS then GRPH (F4) menu, the x defined in the bracket is the variable that the derivative is being found with respect to.

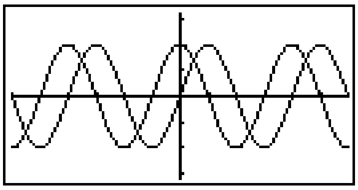


The view-window can be set for trigonometric functions using the menu option of TRIG, the minimum and maximum Y-values will need to be defined. The angle measure must be in radians for the screen capture shown. (see SET UP if degree values appear).

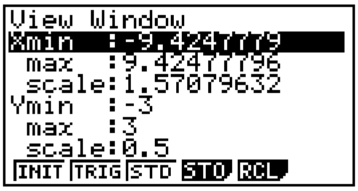
The view-window can be accessed via the function buttons below the screen.

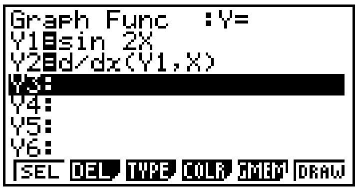


The graphs of $y = \sin x$ and its derivative suggest that the derivative is a cosine function, its amplitude is the same as the original function and its period is also the same as the original function. From this the derivative appears to be $\cos x$.



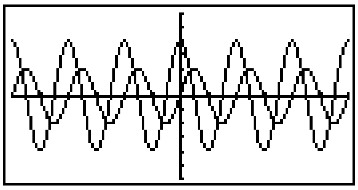
The advantage of the set up used here is that as the function in Y1 is altered, the function in Y2 will always the derivative of Y1. After altering the view window Y1 has been altered to sin 2x and the resulting graphs produced.





Students would be expected to notice that the derivative function is:

(i) cosine in nature
(ii) of the same period as the original function
(iii) twice the amplitude of the original function



This leads to the conclusion that $\frac{d}{dx}(\sin 2x) = 2\cos 2x$

Students would then be asked to suggest a rule and test it using y = sin 3x and from this investigation make a general statement about the derivative of sin kx.

i.e. $\frac{d}{dx}(\sin kx) = k\cos kx$

Students would then complete similar investigations to determine the general forms:

$$\frac{d}{dx}(A\sin kx) = Ak\cos kx$$

$$\frac{d}{dx}(A\cos kx) = -Ak\sin kx$$

The exponential function and its derivative

Level
Upper secondary

Mathematical Ideas
Exponential functions, derivatives, graphical and tabular forms

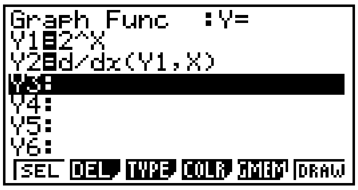
Description and Rationale
This activity may be structured for use as a revision lesson or as a discovery lesson for students. The ability to view information in a number of forms both separately and simultaneously can be a valuable tool in the teaching and learning of mathematics. Here exponential functions and their derivatives are investigated and conclusions on the relationships between the two may be checked. The opportunity to check the differentiation of exponential functions by viewing both graphical and tabular forms is a very powerful teaching tool.

An interesting starting point involves determining what the value of Euler's number (e) actually is. The value of e is defined such that :

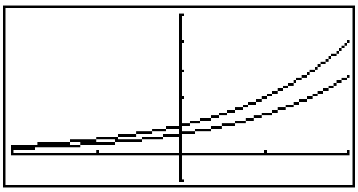
$$\frac{d}{dx}(e^x) = e^x$$

The value of e could be determined by investigating different values of "a" in the form $y = a^x$. By graphing $y = a^x$ and the derivative of $y = a^x$ simultaneously adjustments can be made until the two graphs appear to match, this can then be refined further by continuing these adjustments in the TABLE mode.

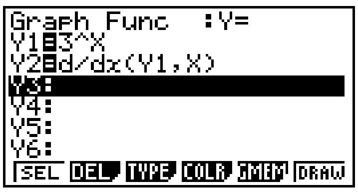
To define the derivative for Y2 the OPTN button then the CALC (F2) menu is used. The defining of the function Y1 is via the VARS then GRPH (F4) menu, the *x* defined in the bracket is the variable that the derivative is with respect to.



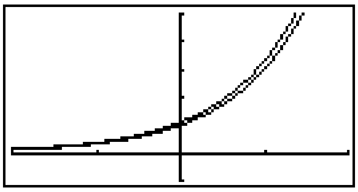
The graph shows the derivative graph below the original function. This suggests the gradient function is less than the original function. The solution is to increase the value of "a".



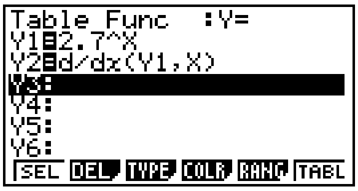
The exponential function is increased to $y = 3^x$.

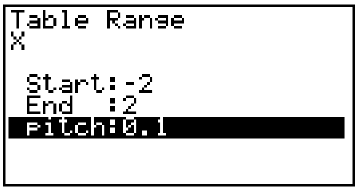


The gradient function is now above the original function, suggesting the value of “a” is now greater than the value being searched for. We know that $2 < e < 3$

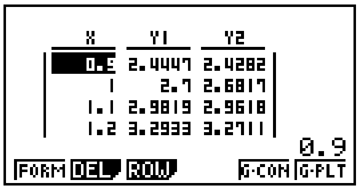


Because the graphs are now very close together the use of the TABLE function would allow for a much more precise exploration for the value of Euler’s number.



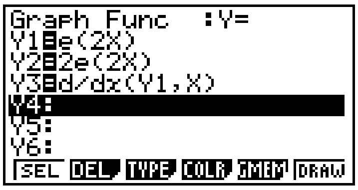


The values in the table show that the gradient function is almost equal to the original function. With continued adjustment a value of “e” to the required accuracy can be obtained.

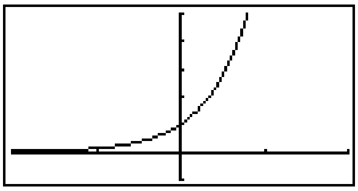


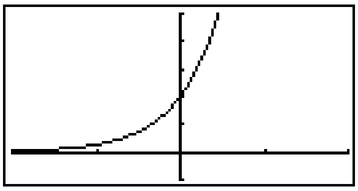
These techniques can be used in a variety ways to investigate, explore and revise exponential functions. An exponential function could be placed into Y1. The students solution to the derivative of this function is placed in Y2, with the automatic derivative placed into Y3 to check the students response. Both GRAPH and TABLE modes may be used in this task.

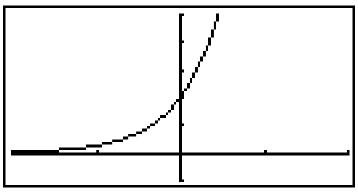
An original function of the form $y = e^{2x}$, the proposed derivative and the actual derivative are shown.



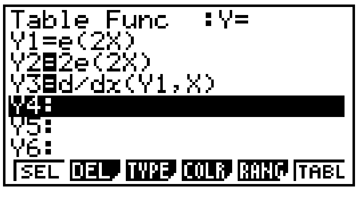
The graphs are usually drawn on the one screen, here they are shown individually for easy comparison.

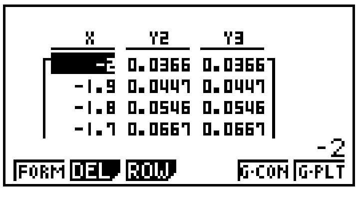




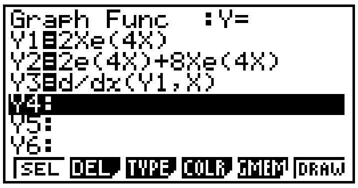


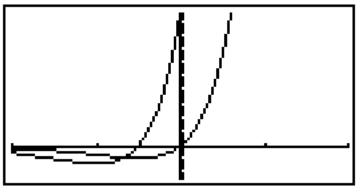
A similar approach with tables may also be employed.

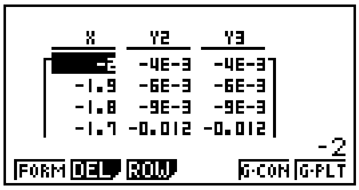




The derivatives of more complex functions can also be explored and checked using either of these approaches.

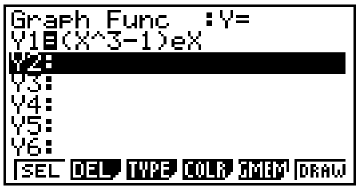




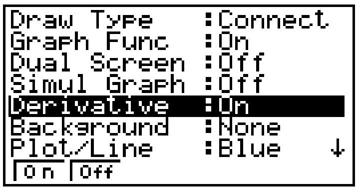


Another useful application involves the sketching of gradient graphs.

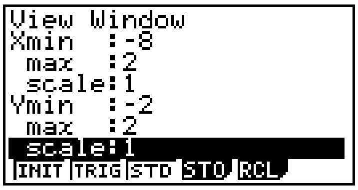
Students could be asked to sketch the gradient function of complex functions with the assistance of the calculator.

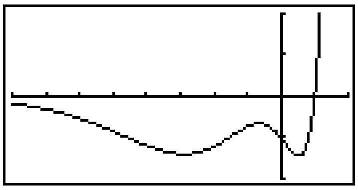


The value of the gradient at any point can be generated by tracing the curve with the DERIVATIVE ON. This can be activated via the SET-UP.

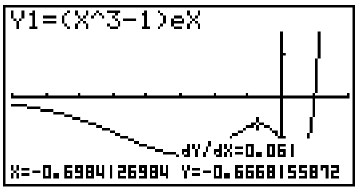


An appropriate view window must be selected and then the function drawn.

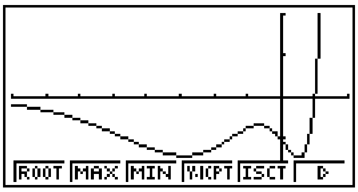




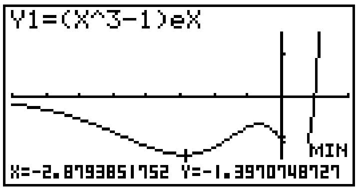
When the TRACE (SHIFT F1) is pressed the x coordinate, y coordinate and the gradient are simultaneously displayed.

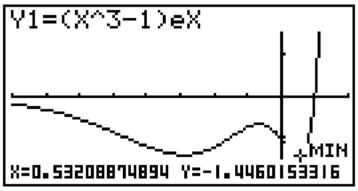


Stationary points can also be calculated using the G-Solv function.

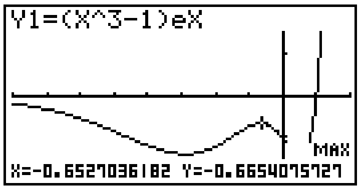


The two local minima can be found by pressing MIN (F3). The second minimum is found by pressing the right arrow.





The local maximum is found by pressing MAX (F2).



This information can then be used to sketch gradient functions. This approach may also be useful for checking the solutions generated by an algebraic approach to similar problems.

Kepler's Law

Level
Upper secondary

Mathematical Ideas
Modelling data, power variation, straightening data with logarithms, residual plots

Description and Rationale
Many traditional mathematics problems have been given a new life with the advent of the graphics calculator. The time spent in completing repetitive calculating tasks can be greatly reduced with the use of a graphics calculator, allowing for a greater focus on the important mathematics involved in the task. This activity allows students to analyse historical data to determine a famous mathematical relationship. The activity may be used at a variety of levels and allows for a variety of approaches. A data set relating periods and orbital radii for a number of planets in our solar system is provided, along with the general accepted form of the model. The model is determined and compared to its accepted form, along with a view of a residual plot to confirm the acceptability of the model.

Johann Kepler is a very famous man in the history of astronomy and mathematics. He used data from observations of planetary orbits to show that these motions were not random, that they in fact obeyed certain mathematical laws, and that these laws could be written in algebraic form. The data below uses the earth as the base unit, so times and distances are given as multiples of 1 earth year, and the average radius of our Earth's orbit around the sun.

Some of his observational data are given below:

Planet	Orbital period T	Orbital radius R
Mercury	0.241	0.387
Venus	0.615	0.723
Earth	1.000	1.000
Mars	1.881	1.542
Jupiter	11.862	5.202
Saturn	29.457	9.539

Kepler showed that a power model could be used to describe this data.

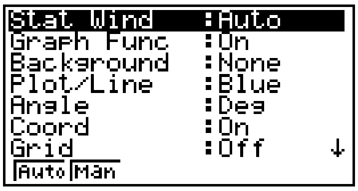
i.e. $T = aR^b$ where a and b are constants.

The task is to find appropriate values for a and b, and to justify the procedure you use.
A follow up task involves reorganising the solution into the traditional expression of the law.

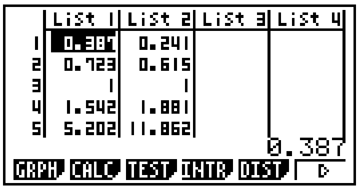
This law is usually stated in the form $T^m = k R^n$, where m and n are whole numbers. Determine the simplest values of m and n and complete Kepler's law.

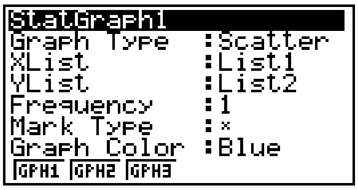
> "…The __________________ of a planet's orbital period (T in years) is
>
> directly proportional to the __________________ of the average distance
>
> (R in metres) from the sun…"

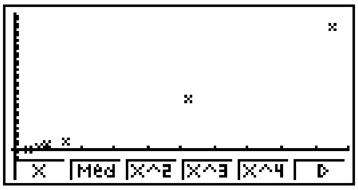
For an easy scatter plot the AUTO window needs to be defined. This is done in the SET UP mode (SHIFT then MENU).



The data needs to be entered in the STAT mode and plotted.



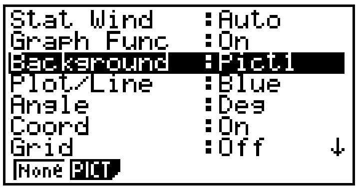




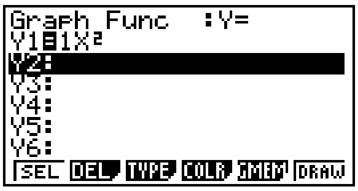
The formulation of an appropriate model may be completed in a number of ways.

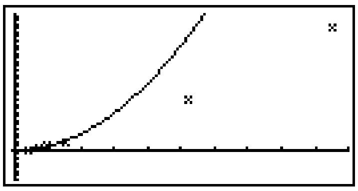
Students may be asked to develop the model by appropriately selecting and adjusting values for the two constants *a* and *b* until a satisfactory model is produced.

The first step is to place the scatter plot in the background of the graphing screen. A picture is taken with the scatter plot on the screen via the OPTN then PIC menus. This is placed in the background by going into the SET UP (SHIFT then MENU) and making the saved picture the background



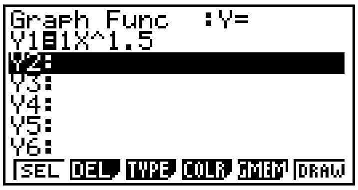
A starting model needs to be created as shown. The curved nature of the scatter plot suggesting a starting model as shown.

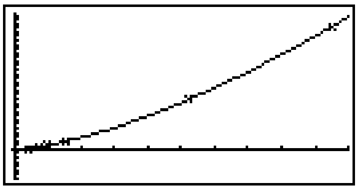




The scatter plot produced suggests that the power of 2 is too large, a value less than 2 (1.5) needs to be tested.

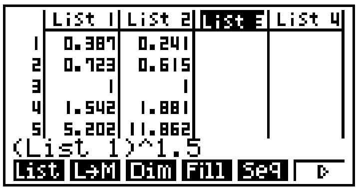
The new model and resulting scatter plot are shown below.



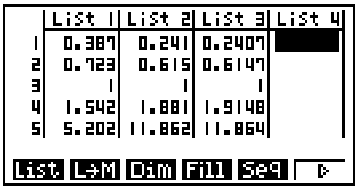


This would appear to be suitable model of the data provided. The value of *a* can be analysed by returning to the lists and completing the operation of the rule on the values in List 1 and then comparing List 2 (actual data) and List 3 (predicted values).

To operate on a list of numbers the list name is highlighted. The available list commands are found via the OPTN button then LIST menu.

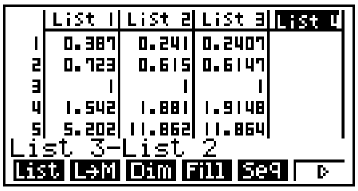


The resulting table of actual and predicted data can then be compared.

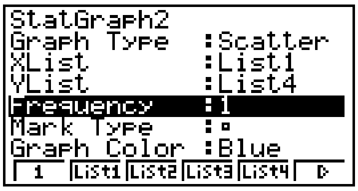


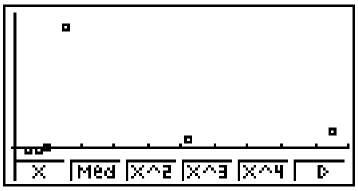
A method of determining the quality of the model is to do a Residual plot. This involves looking for a pattern with the differences between the actual data and the results predicted by the model. A plot where the residuals appear to be randomly distributed supports the proposed model.

The residuals may be calculated by defining the operation as shown.



The set up of the scatter plot and the resulting plot are shown below.





The variations from the proposed model are small in size although not completely randomly distributed.

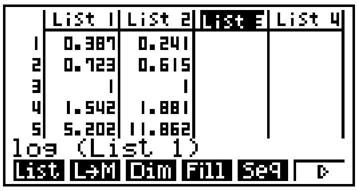
A second approach to this task would involve attempting to straighten the data using logarithms. Natural logarithms or logarithms to the base 10 may be used.

If the proposed model is $T = a R^b$ then :

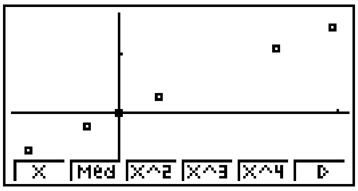
$$\log T = \log (a R^b)$$
$$\log T = \log a + \log R^b$$
$$\log T = \log a + b \log R$$
$$\log T = b \log R + \log a$$

This is in the general form of a straight line (y = mx + c) and so if the model is appropriate the data above may be modelled by a straight line if plotted as log T vs log R. The gradient of this plot gives the value for b, while the y-intercept provides the value of log a.

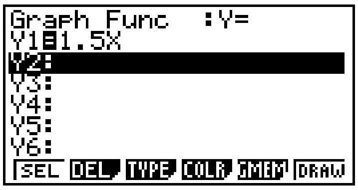
The commands to find the logarithms of List 1 are shown. The commands are accessed via the OPTN button and LIST menu.

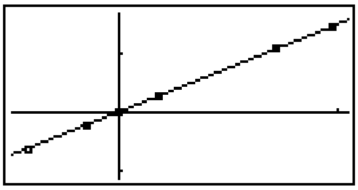


The scatter plot of log T vs log R is shown.



A model may be fitted as before or by using the regression potential of the calculator.





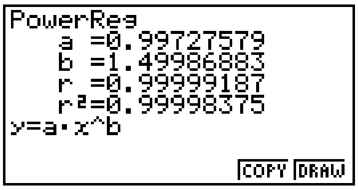
This model must now be reconstructed into the required form.

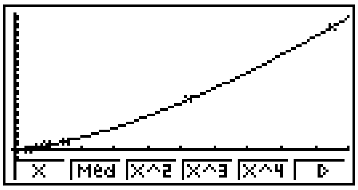
$$\log T = 1.5 \log R + 0$$
$$\log a = 0, \quad a = 1, \ n = 1.5$$
$$T = 1 \, . R^{1.5}$$

The third method uses the regression model potential of the calculator, while this is the least mathematically rich option for the student there are times when this method would be used. The added bonus is that the residual analysis can easily be completed.

In the STAT mode the residuals can be stored in a list.
In the SET UP mode (SHIFT then MENU) the residuals have been placed into list 3.

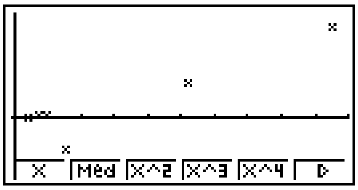
The scatter plot can be produced, the power model (Pwr) generated and the model placed over the data.





The calculator has placed the residuals placed into List 3.

A scatter plot of the residuals (List 3) vs the orbital radii (List 1) can easily be produced for investigation.



A model of the form $T = R^{1.5}$ has been generated by each of the methods employed. The second part of the task involved determining the general form of Kepler's law.

If the power is 1.5 then if both sides are squared the model becomes $T^2 = R^3$. The general statement of Kepler's law becomes:

> "…The **square** of a planet's orbital period (T in years) is directly proportional to the **cube** of the average distance (R in metres) from the sun…"

This is in fact a result that can be determined from Newton's Law of Gravitation. Newton's Law states that the force between two objects is directly proportional to the product of the two masses (m_1 and m_2) and inversely proportional to the square of their separation (R).

$F = \frac{Gm_1m_2}{R^2} = ma$ where G is the Universal Gravitation constant, 6.67×10^{-11} $Nm^2 kg^{-2}$

For orbiting bodies this acceleration a is $\frac{v^2}{R}$, $v = \frac{2\pi R}{T}$ therefore $a = \frac{4\pi^2 R}{T^2}$

Subbing this into the above equation results in $\frac{R^3}{T^2} = \frac{GM}{4\pi^2}$, where M is the mass of the central body (the sun).

The right hand side of the above expression is the constant of proportionality, dependent only on the mass of the central body of the system.

The Love Bug virus

Level
Upper secondary

Mathematical Ideas
Exponential functions, modelling real data

Description and Rationale
The recent havoc wreaked in cyber-world by the infamous Love Bug (computer virus) has served as the inspiration for the following technology-based activity. It involves a simulation of students exchanging e-mail messages, in order to investigate the spread of a computer virus if one "computer" is initially infected. The task is designed to illustrate a real-world instance of exponential growth, a topic that is usually found in senior secondary mathematics courses. Students should be encouraged to comment on the appropriateness of the exponential model and design further experiments that more closely match the process of sending, opening, and replying to e-mails.

Resources
A Styrofoam cup for each student
Red Litmus Paper
Class set of graphics calculators and overhead projector unit or VI video link
2M NaOH (sodium hydroxide) solution (see the science technician about access to an appropriately concentrated sample of sodium hydroxide)

The Activity
1. Before the lesson, half fill one cup with the NaOH solution and half fill the rest of the cups with water.
2. Distribute the cups randomly to each student when the lesson begins.
3. In pairs, the students should "exchange e-mail messages" with each other. That is, one student pours his/her liquid into the other student's cup (sends a message) and then the second student pours half the mixed liquid back into the first student's cup (replies to the message).
4. Students are to then line up for their 'virus check'. That is, the teacher tests each student's cup with the red litmus paper for the presence of base (NaOH) and results are recorded in the following table. (A data logger with pH probe can be used, if available, to check for the presence of the 'virus'. However, the litmus paper colour change offers greater visual impact.)

Number of E-mail Exchanges	*Number of Computers Infected*
0	1

Note: Initially, there is only one computer infected.

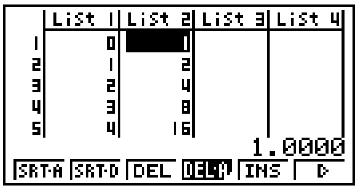
5. The students are to exchange another 'e-mail message' with a different person and repeat steps 3 and 4.

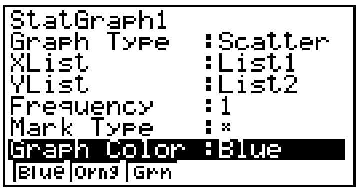
6. Repeat step 5 until all the students' computers in the class are infected with the virus, that is, the base (NaOH) is present in all the students' cups.

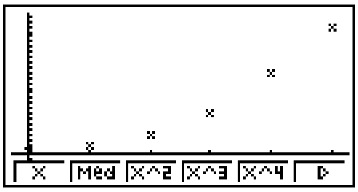
The following is a typical set of data for this activity, based on a class size of 25 students.

No. of e-mail exchanges	No. of computers infected
0	1
1	2
2	4
3	8
4	16
5	25

The data is placed into the lists via the STAT mode. A scatterplot is defined and drawn via the GRPH(F1) and SET(F6) commands.



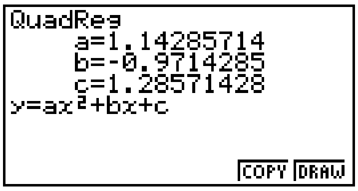


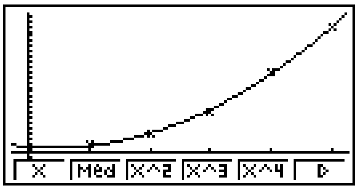


There are a number of possible models, these include: quadratic
exponential
logistic

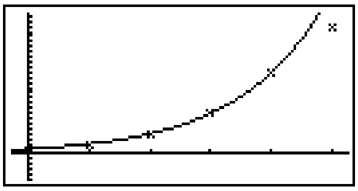
Each of these models produces a satisfactory model over the range of the data collected, however the long term viability of such an approach need to be considered individually.

The quadratic model, determined here by the regression command, is shown below. While it fits the data well over the range of the data the nature of the squaring is that the growth in infection would not match the rapid spreading of the disease.

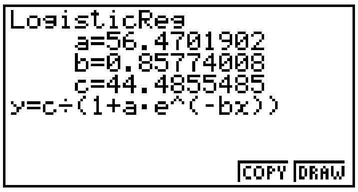


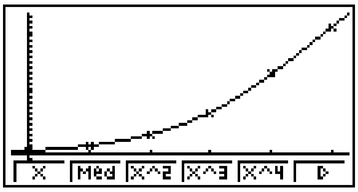


The exponential model would expected to be of the form $y = 2^x$, provided there was not a significant number of cases where infected computers were communicating. This is reasonable when the number of infections is much less than the number of computers. As the number of infections increases and approaches the number of computers in the system the less effective will be the model.

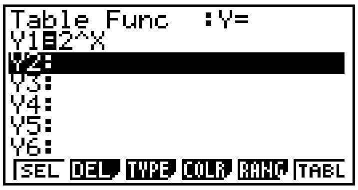


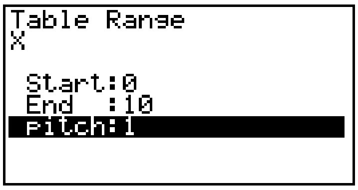
A more appropriate model in this case may be a logistic function. It shows the rapid growth in infections early on in the process along with the slowing down as the number of infections increases and the situation where infected computers have more exchanges with other infected computers occurs. The exact details of such a model are not best explored in this situation.

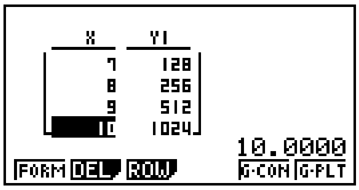




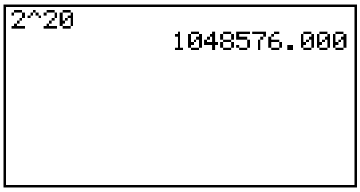
Assuming the exponential model students could be asked to make predictions and answer a number of questions.
A question such as “How many e-mail exchanges would need to occur in order to infect 100 computers with the virus? 1000 computers with the virus?” can best be answered using the TABLE function capabilities. The rule is placed in the table command and the range is set to give an appropriate amount of data.
The 100th infection occurs during the 7th exchange, while the 1000th exchange occurs during the 10th exchange.







Students could be asked to predict the number of computers infected after 20 e-mail exchanges. This can easily be calculated in the RUN mode.



One of the areas where students need to improve is in the communication of mathematics. This ranges from explaining the approach and solution to a task to discussing the key issues in the construction and completion of a modelling activity. Students could be asked to consider the effect of class size on the determination of the model, and could lead to discussion of the effectiveness of the simulation in obtaining a model for the real-life situation.
A more general discussion on identifying limitations of models in the light of the real world context may follow. One major consideration revolves around do people always reply to e-mail messages, especially those from unknown senders. If not then a rethinking of our simulation model is necessary.

This activity is adapted from a similar task that provides a demonstration of the transmission of the AIDS virus (Legere & Stringer). For the original task the mixing of water and NaOH provided a realistic simulation of the exchange of body fluids through which the AIDS virus is transmitted. However, the spread of computer viruses is a little different, since the "exchange" of e-mail messages can be clearly separated into "sending" and "receiving". In the present activity this difficulty is avoided by having students both "send" and "reply" to messages, thus allowing an already infected receiver to pass the virus to the sender in their reply. Students should be encouraged to evaluate the appropriateness of the exponential model in these circumstances, and to suggest how the simulation might be modified to more closely match the behaviour of e-mail users (e.g. not opening or replying to an infected message can be represented by not pouring half the mixed liquid back into the first student's cup).

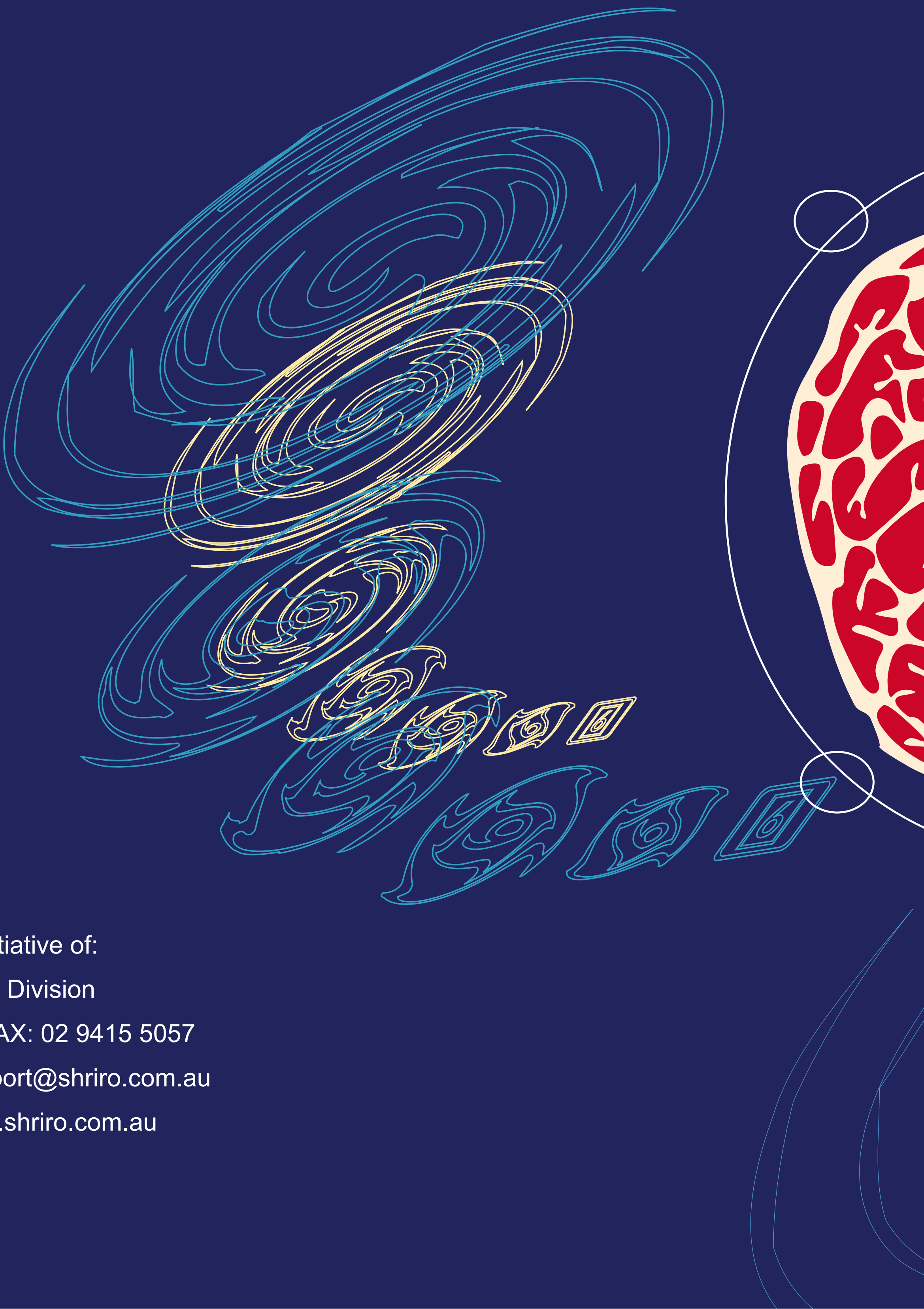
This activity is quite open-ended, and has the potential to be used in a number of different classroom situations. The following list outlines some possible modifications and extensions in addition to that mentioned above:

- The activity could be used as an assessment task, with data collection done in class.
- Investigate the data obtained if two different viruses are spreading simultaneously.
- Investigate the effect on data if one or more student/s are in possession of a "virus scanner", i.e. a piece of litmus paper used to test **before** "receiving an e-mail".
- Investigate ways of altering the rate at which the virus spreads. Students could discuss this from various viewpoints e.g. creators of the virus versus a company who lost billions of dollars from its effects.

Reference

Legere, C. & Stringer, C. AIDS transmission demonstration.

The learning of many mathematical concepts can occur in new ways and traditional approaches can be enhanced with the use of graphics calculator technology. This book offers ideas on how a graphics calculator can be used appropriately in the learning and doing of mathematics. The lesson ideas included in this book have been submitted by teachers from across Australia.



CASIO

This product is an initiative of:
The Casio Education Division
Ph: 02 9415 5052 FAX: 02 9415 5057
email: casio.edusupport@shriro.com.au
http://www.casio.edu.shriro.com.au

ISBN: 0 9578302 1 1